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ON (b_i)-MORPHISMS OF ALGEBRAS

BY

GH. COSTOVICI

1. In this Note we establish some results about (b_i)-morphisms of algebras here defined. We mention that when the finite bijection (b_i) is the identity, an (b_i)-morphism becomes homomorphism.

2. Suppose that X and Y are algebras equipped with the operations $f_i : X^{n_i} \to X$ and $g_i : Y^{n_i} \to Y$, $n_i \in \mathbf{N} =$ the set of the positive integers, $i \in I =$ arbitrary set. (b_i) denotes the bijection $(i_1, i_2, \ldots, i_{n_i})$ of $(1, 2, \ldots, n_i)$, $n_i \in \mathbf{N}$, $i \in I$. The mapping $H : X \to Y$ is (b_i)-morphism relative to the operation f_i, $i \in I$ $\Leftrightarrow \forall x_j \in X$, $j = \overline{1, n_i}$, $H(f_i(x_1, x_2, \ldots, x_{n_i})) = g_i(H(x_{i_1}), H(x_{i_2}), \ldots, H(x_{i_{n_i}}))$. The equivalence ρ on X is congruence on $X \Leftrightarrow$ for every $i \in I$ and $\forall x_j, z_j \in X, j = \overline{1, n_i}$ so that $x_j \rho z_j$, $j = \overline{1, n_i}$ then $f_i(x_1, x_2, \ldots, x_{n_i}) \rho f_i(z_1, z_2, \ldots, z_{n_i})$.

3. Teorem 1. *Suppose that X and Y are algebras equipped with the operations $f_i : X^{n_i} \to X$ and $g_i : Y^{n_i} \to Y$, $n_i \in \mathbf{N}$, $i \in I$. Suppose that H_1 and $H_2 : X \to Y$ are, for every $i \in I$, (b_i)-morphisms relative to the operation f_i where $(b_i) = (i_1, i_2, \ldots, i_{n_i})$ as previously. Then the set $S = \{x \in X : H_1(x) = H_2(x)\}$ is subalgebra of X.*

Proof. Let $i \in I$ and $x_j \in S$ be, $j = \overline{1, n_i} \Leftrightarrow H_1(x_j) = H_2(x_j)$, $j = \overline{1, n_i}$. We have $H_1(f_i(x_1, x_2, \ldots, x_{n_i})) = g_i(H_1(x_{i_1}), H_1(x_{i_2}), \ldots, H_1(x_{i_{n_i}})) = g_i(H_2(x_{i_1}), H_2(x_{i_2}), \ldots, H_2(x_{i_{n_i}})) = H_2(f_i(x_1, x_2, \ldots, x_{n_i}))$. Therefore $\forall i \in I$ and $\forall x_j \in S, j = \overline{1, n_i}$, we have $f_i(x_1, x_2, \ldots, x_{n_i}) \in S$ and so S is subalgebra of X.

Theorem 2. *Suppose that X is an algebra equipped with the operations $f_i : X^{n_i} \to X$, $i = \overline{1, k}$, $n_i \in \mathbf{N}$. Suppose that $H : X \to X$ is, for every $i = \overline{1, k}$, (b_i)-morphism relative to the operation f_i, where $(b_i) = (i_1, i_2, \ldots, i_{n_i})$. Denote with $o_i =$ the order of (b_i), $i = \overline{1, k}$, and $c =$ l. c.m. of o_i. Then $H^{nc+1} : X \to X$, $n \in \mathbf{N}^* = \mathbf{N} \cup \{0\}$ is, for every $i = \overline{1, k}$, (b_i)-morphism relative to f_i, as H is.*

Proof by induction on n. For $n = 0$ we have $H : X \to X$ (b_i)-morphism relative to f_i, $i = \overline{1, k}$. Suppose that $H^{nc+1} : X \to X$ is (b_i)-morphism relative to f_i, $i = \overline{1, k}$, so that the following condition is fulfilled ; for every $i = \overline{1, k}$ and $\forall x_j \in X$, $j = \overline{1, n_i}$, we have

$$H^{nc+1}(f_i(x_1, x_2, \ldots, x_{n_i})) = f_i(H^{nc+1}(x_{i_1}), H^{nc+1}(x_{i_2}), \ldots, H^{nc+1}(x_{i_{n_i}})).$$

We are interested of

$$H^{(n+1)c+1}(f_i(x_1, x_2,\dots, x_{n_i})) = H^{nc+1}(H^c(f_i(x_1, x_2,\dots, x_{n_i}))) =$$

$$= H^{nc+1}(f_i(H^c(x_1), H^c(x_2),\dots, H^c(x_{n_i}))) =$$

$$= f_i(H^{nc+1}(H^c(x_{i_1})), H^{nc+1}(H^c(x_{i_2})),\dots, H^{nc+1}(H^c(x_{i_{n_i}}))) =$$

$$= f_i(H^{(n+1)c+1}(x_{i_1}), H^{(n+1)c+1}(x_{i_2}),\dots, H^{(n+1)n+1}(x_{i_{n_i}}))$$

and so $H^{(n+1)c+1}$ is (b_i)-morphism relative to f_i.

Corollary 1. *Let X be an algebra equipped with the operations $f_i : X^{n_i} \to X$, $n_i \in \mathbf{N}$, $i \in I$ and let $H : X \to X$ be a (b_i)-morphism relative to f_i with $(b_i) = (1, 2,\dots, n_i)$ for every $i \in I$. Then the set $\sigma_n = \{x \in X : H^n(x) = H^{n-1}(x)\}$, $n \in \mathbf{N}$ with $H^0(x) = 1_X(x) = x$, is a subalgebra of X and we have $\sigma_1 \subseteq \sigma_2 \subseteq \dots \subseteq \sigma_n \subseteq \dots \subseteq X$.*

The proof follows from Theorem 1.

Corollary 2. *Let X be an algebra equipped with the operations $f_i : X^{n_i} \to X$, $n_i \in \mathbf{N}$, $i = \overline{1, k}$ and let $H : X \to X$ be, for every $i = \overline{1, k}$ a (b_i)-morphism relative to f_i with $(b_i) = (i_1, i_2,\dots, i_{n_i})$. Then, with the notations from Theorem 2, the set*

$$S_{nc+1} = \{x \in X ;\ H^{(n+1)c+1}(x) = H^{nc+1}(x)\},\quad n \in \mathbf{N}^* = \mathbf{N} \cup \{0\},$$

is a subalgebra of X and we have

$$S_1 \subseteq S_{c+1} \subseteq S_{2c+1} \subseteq \dots \subseteq S_{nc+1} \subseteq \dots \subseteq X.$$

The proof follows from the Theorems 1 and 2.

Theorem 3. *Suppose that X and Y are algebras equipped with the operations $f_i : X^{n_i} \to X$ and $g_i : Y^{n_i} \to Y$, $n_i \in \mathbf{N}$, $i \in I$. Suppose that $H : X \to Y$ is, for every $i \in I$, (b_i)-morphism relative to f_i with $(b_i) = (i_1, i_2,\dots, i_{n_i})$. If we define ρ on X by $x \rho z \Leftrightarrow H(x) = H(z)$, then we have :*

1° *ρ is a congruence on X ;*

2° *X/ρ becomes an algebra by the operations $\overline{f}_i : (X/\rho)^{n_i} \to X/\rho$ where $\overline{f}_i(\overline{x}_1, \overline{x}_2,\dots, \overline{x}_{n_i}) = \overline{f_i(x_1, x_2,\dots, x_{n_i})}$;*

3° *$\overline{H} : X/\rho \to Y_1 = H(X)$ defined by $\overline{H}(\overline{x}) = H(x)$ is, for every $i \in I$, (b_i)-morphism bijective relative to $\overline{f}_i$ with $(b_i) = (i_1, i_2,\dots, i_{n_i})$.*

Proof. 1. Evidently, ρ is equivalence on X. If $i \in I$ and $x_j \rho z_j$, $j = \overline{1, n_i}$, then $H(x_j) = H(z_j)$, $j = \overline{1, n_i}$. We have $H(f_i(x_1, x_2,\dots, x_{n_i})) = g_i(H(x_{i_1}), H(x_{i_2}),\dots, H(x_{i_{n_i}})) = g_i(H(z_{i_1}), H(z_{i_2}),\dots, H(z_{i_{n_i}})) = H(f_i(z_1, z_2,\dots, z_{n_i}))$ so that $f_i(x_i, x_2,\dots, x_{n_i})$ $\rho f_i(z_1, z_2,\dots, z_{n_i})$ and therefore ρ is congruence.

2. For every $i \in I$, $\overline{f}_i$ is well defined because if $z_j \rho x_j$, $j = \overline{1, n_i}$ then $H(z_j) = H(x_j)$, $j = \overline{1, n_i}$ and $H(f_i(x_1, x_2,\dots, x_{n_i})) = g_i(H(x_{i_1}), H(x_{i_2}),\dots, H(x_{i_{n_i}})) = g_i(H(z_{i_1}), H(z_{i_2}),\dots, H(z_{i_{n_i}})) = H(f_i(z_1, z_2,\dots, z_{n_i}))$ and then $f_i(x_1, x_2,\dots, x_{n_i})$ $\rho f_i(z_1, z_2,\dots, z_{n_i}) \Leftrightarrow \overline{f_i(x_1, x_2,\dots, x_{n_i})} = \overline{f_i(z_1, z_2,\dots, z_{n_i})} \Leftrightarrow \overline{f}_i(\bar{x}_1, \bar{x}_2,\dots, \bar{x}_{n_i}) = \overline{f}_i(\bar{z}_1, \bar{z}_2,\dots, \bar{z}_{n_i})$.

3. $\bar{H}(\bar{x})=\bar{H}(\bar{z})\Leftrightarrow H(x)=H(z)\Leftrightarrow x\rho z\Leftrightarrow \bar{x}=\bar{z}$ and so $\bar{H}$ is injection. If $y=$ $=H(x)\in Y_1=H(X)$, $\exists\bar{x}\in X/\rho$ so that $\bar{H}(\bar{x})=H(x)=y$ and then $\bar{H}$ is surjection. For every $i\in I$ we have

$$\bar{H}(\bar{f}_i(\bar{x}_1, \bar{x}_2,\dots, \bar{x}_{n_i}))=\bar{H}(\overline{f_i(x_1, x_2,\dots, x_{n_i})})=H(f_i(x_1, x_2,\dots, x_{n_i}))=$$

$$=g_i(H(x_{i_1}), H(x_{i_2}),\dots, H(x_{i_{n_i}}))=g_i((\bar{H}(\bar{x}_{i_1}), \bar{H}(\bar{x}_{i_2}),\dots, \bar{H}(\bar{x}_{i_{n_i}}))$$

and then $\bar{H}$ is (b_i)-morphism relative to $\bar{f}_i$ with $(b_i)=(i_1, i_2,\dots, i_{n_i})$.

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DESPRE (b_i)-MORFISME DE ALGEBRE

(Rezumat)

Se introduce noţiunea de (b_i)-morfism între algebre cu operaţii finitare, şi se stabilesc unele rezultate. Cînd bijecţia (b_i) este identitatea, un (b_i)-morfism este homomorfism.

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GAUGE THEORY ON A VECTOR BUNDLE

IV. AN APPLICATION TO THE ELECTROMAGNETIC FIELD THEORY

BY

AUREL BEJANCU

0. Introduction. This is the fourth paper in a series we devoted to the gauge theory on a vector bundle. We recall that in the first paper [1] we developed o relativistic Finsler geometry on a vector bundle and studied the global gauge invariance of Lagrangians. The second paper of this series was concerned with local gauge invariance of Lagrangians on a vector bundle. Moreover, we obtained three types of Lagrangians for horizontal and vertical gauge fields on a vector bundle. Finally, in the third paper we defined the strength fields and obtained the equations of motion, the conservation laws and the Bianchi identities for a gauge theory on a vector bundle.

It is the purpose of the present paper to give an application of the mathematical apparatus we developed in our previous papers [1]...[3]. More precisely, we consider on the total space of a vector bundle over the Minkowski space the Lagrangian (1.11) which is invariant with respect to the phase transformations Lie group $U(1)$. In this was we obtain a generalization of the electromagnetic field theory. It is remarkable that the generalized Dirac-Maxwell Eqs. (1.36) are obtained from the classical ones by a simple replacing of partial derivatives $\frac{\partial}{\partial x^\alpha}$ by the operators $\frac{\delta}{\delta x^\alpha}$ given by (1.7).

1. An Electromagnetic Field Theory Based on a Gauge Theory on a Vector Bundle. As it is well-known, the classical electromagnetic field theory follows from a gauge theory with respect to the phase transformations Lie group $U(1)$ and a certain Lagrangian constructed by means of some spinor fields (see Chaichian-Nelipa [4]). For this reason, we define in the first part of the paper, spinor fields on a vector bundle over a Minkowski space and show that the mathematical apparatus we developed in [1]...[3] still works in this case.

Let M be the 4-dimensional Minkowski space endowed with a flat pseudo-Riemannian metric $\eta=(\eta_{\alpha\beta})$ of Lorentz signature (1, 3), i.e., we have

$$\eta_{\alpha\beta}=\begin{cases} 1 & \alpha=\beta=1,2,3 \\ -1 & \alpha=\beta=0 \\ 0 & \alpha\neq\beta. \end{cases} \tag{1.1}$$

Suppose E is the total space of a vector bundle ξ over M with m-dimensional fibers. The *relativistic transformations* on E are given by

$$\begin{cases} \tilde{x}^\alpha = L^\alpha_\beta x^\beta + A^\alpha \\ \tilde{v}^i = A^i_j(x) v^j, \end{cases} \tag{1.2}$$

where L^α_β satisfy

$$L^\alpha_\beta L^\varepsilon_\mu \eta_{\alpha\varepsilon} = \eta_{\beta\mu}, \tag{1.3}$$

A^α are real constants and $A^i_j(x)$ are differentiable functions locally defined on E.

Next, we denote by γ_α the *Dirac matrices* and define

$$\sigma_{\alpha\beta} = \frac{1}{2i}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha). \tag{1.4}$$

Then we say that

$$\Phi(x, v) = \begin{bmatrix} \Phi_1(x, v) \\ \Phi_2(x, v) \\ \Phi_3(x, v) \\ \Phi_4(x, v) \end{bmatrix} \tag{1.5}$$

is a *four components Dirac spinor field* (*Dirac wave function*) if with respect to the relativistic transformations (1.2) we have

$$\tilde{\Phi}(\tilde{x}, \tilde{y}) = \exp(i\varepsilon^{\alpha\beta}\sigma_{\alpha\beta})\Phi(x, v), \tag{1.6}$$

where $\varepsilon^{\alpha\beta}$ are the infinitesimal angles of rotation.

Suppose now $HE = (N^i_\alpha(x, v))$ is a non-linear connection on E and consider the operators

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - N^i_\alpha(x, v)\frac{\partial}{\partial v^i}. \tag{1.7}$$

Usually, the operators from (1.7) are acting on differentiable scalar fields locally defined on E. We examine now the possibility of expanding their action to Dirac spinor fields.

First, by (1.6) and (1.7) of [1] we have

$$k\frac{\partial\Phi}{\partial x^\alpha} = k\left(L^\beta_\alpha\frac{\partial\Phi}{\partial\tilde{x}^\beta} + \frac{\partial A^i_j(x)}{\partial x^\alpha}v^j\frac{\partial\Phi}{\partial\tilde{v}^i}\right) = L^\beta_\alpha\frac{\partial\tilde{\Phi}}{\partial\tilde{x}^\beta} + \frac{\partial A^i_j(x)}{\partial x^\alpha}v^j\frac{\partial\tilde{\Phi}}{\partial\tilde{v}^i}, \tag{1.8}$$

and respectively

$$k\frac{\partial\Phi}{\partial v^i} = kA^j_i(x)\frac{\partial\Phi}{\partial\tilde{v}^j} = A^j_i(x)\frac{\partial\tilde{\Phi}}{\partial\tilde{v}^j}, \tag{1.9}$$

where we put $k = \exp(i\varepsilon^{\alpha\beta}\varepsilon\sigma_{\alpha\beta})$. Thus by using (1.16) from [1] and (1.7)—(1.9) from the present paper we obtain

$$k\frac{\delta\Phi}{\delta x^\alpha} = L^\beta_\alpha\frac{\delta\tilde{\Phi}}{\delta\tilde{x}^\beta}. \tag{1.10}$$

Next we may follow the same lines as in [1] and obtain the same equations of motion (2.25) of [1]. Moreover, the local gauge invariant Lagrangian $\mathcal{L}_0(x, y)$ is obtained by replacing the derivatives $\frac{\delta\Phi}{\delta x^\alpha}$ and $\frac{\partial\Phi}{\partial v^i}$ by $\overset{(h)}{D}_\alpha\Phi$ and $\overset{(v)}{D}_i\Phi$ given by (1.17) and (1.18) of [2] respectively. Hence all our results from [1]...[3] can be considered for Lagrangians constructed by means of Dirac spinor fields too.

Now, let $\Phi_i(x, v)$ a Dirac spinor on field E and

$$\bar{\Phi}(x, v) = [\Phi_1^*(x, y) \quad \Phi_2^*(x, v) \quad -\Phi_3^*(x, v) \quad -\Phi_4^*(x, v)]$$

the *Dirac conjugate spinor field*, where $\Phi_i^*(x, v)$ are the complex conjugate fields of $\Phi_i(x, v)$. Then we consider the following Lagrangian

$$\mathcal{L}_0(x, v) = i\eta^{\gamma\beta}\bar{\Phi}\gamma_\alpha \frac{\delta\Phi}{\delta x^\beta} - m\bar{\Phi}\Phi. \tag{1.11}$$

By using (1.3), (1.6) and (1.10) it is easy to check that $\mathcal{L}_0(x, v)$ is invariant with respect to the relativistic transformations (1.2). Moreover, $\mathcal{L}_0(x, v)$ is invariant with respect to the global action of the phase transformations Lie group $U(1)$ given by

$$\Phi \to \Phi' = e^{-ig\varepsilon}\Phi; \quad \bar{\Phi} \to \bar{\Phi}' = e^{ig\varepsilon}\bar{\Phi}, \tag{1.12}$$

where ε is the parameter of $U(1)$ and g is the so called *coupling constant*.

In order to keep the same notations as in the general gauge theory on E from [1]...[3] we put

$$Q^1(x, v) = \Phi(x, v); \quad Q^2(x, v) = \bar{\Phi}(x, v). \tag{1.13}$$

In this case we have : $a=1$; $A, B=1, 2$; $\varepsilon^1=\varepsilon$, $[X_1]^A_B = X^A_B$. Then (2.34) of [1] becomes

$$\begin{bmatrix} \delta Q^1 \\ \delta Q^2 \end{bmatrix} = \varepsilon \begin{bmatrix} X_1^1 & X_2^1 \\ X_1^2 & X_2^2 \end{bmatrix} \times \begin{bmatrix} Q^1 \\ Q^2 \end{bmatrix}. \tag{1.14}$$

By (1.12) we get

$$\delta Q^1 = -ig\varepsilon Q^1; \quad \delta Q^2 = ig\varepsilon Q^2. \tag{1.15}$$

Hence, due to (1.14) and (1.15) we obtain

$$X_2^1 = X_1^2 = 0; \quad X_1^1 = -ig; \quad X_2^2 = ig. \tag{1.16}$$

We now consider the local action of $U(1)$ on $\mathcal{L}_0(x, v)$, that is, the parameter ε becomes now a differentiable function $\varepsilon(x, v)$. Then according to our theory in [2] we have to consider two gauge fields $H_\alpha(x, v)$ and $V_i(x, v)$ which we call the *horizontal* and respectively *vertical electromagnetic fields* on the total space E of a vector bundle. Since in this case we have $C_b{}^a{}_c = 0$, the Eqs. (1.29) and (1.30) from [2] become

$$\overset{*}{\delta}(H_\alpha(x, v)) = \frac{\delta\varepsilon(x, v)}{\delta x^\alpha}, \tag{1.17}$$

and respectively

$$\overset{*}{\delta}(V_i(x, v)) = \frac{\partial\varepsilon(x, v)}{\partial v^i}. \tag{1.18}$$

On the other hand, by using the general formulas (1.17) and (1.18) from [2] of horizontal and respectively vertical gauge covariant derivatives we obtain

$$(1.19)\qquad \begin{cases} \overset{(h)}{D}_\alpha Q^1 = \dfrac{\delta Q^1}{\delta x^\alpha} + igH_\alpha Q^1, \\ \overset{(h)}{D}_\alpha Q^2 = \dfrac{\delta Q^2}{\delta x^\alpha} - igH_\alpha Q^2, \end{cases}$$

and respectively

$$(1.20)\qquad \begin{cases} \overset{(v)}{D}_i Q^1 = \dfrac{\partial Q^1}{\partial v^i} + igV_i Q^1, \\ \overset{(v)}{D}_i Q^2 = \dfrac{\partial Q^2}{\partial v^i} - igV_i Q^2. \end{cases}$$

Therefore, we get a new Lagrangian $\mathcal{L}_0'(x, v)$ which is invariant to the local gauge transformations by replacing $\dfrac{\delta Q^A}{\delta x^\alpha}$ from $\mathcal{L}_0(x, v)$ by $\overset{(h)}{D}_\alpha Q^A$ given by (1.19). Then by a simple calculation we infer

$$(1.21)\qquad \mathcal{L}_0'(x, v) = \mathcal{L}_0(x, v) + \mathcal{L}_I(x, v),$$

where $\mathcal{L}_I(x, v)$ is the interaction Lagrangian for the fields $\bar{\Phi}(x, v)$, $\Phi(x, v)$ and the horizontal electromagnetic field $H_\alpha(x, v)$. More precisely, we have

$$(1.22)\qquad \mathcal{L}_I(x, v) = -g\eta^{\alpha\beta}\bar{\Phi}\gamma_\alpha\Phi H_\beta.$$

Further, according to (2.38), (2.39) and (2.40) from [2] we obtain the following strength fields

$$(1.23)\qquad R_{\alpha\beta} = \frac{\delta H_\alpha}{\delta x^\beta} - \frac{\delta H_\beta}{\delta x^\alpha} + R^i_{\alpha\beta} V_i,$$

$$(1.24)\qquad P_{\alpha i} = \frac{\partial H_\alpha}{\partial v^i} - \frac{\delta V_i}{\delta x^\alpha} + \frac{\partial N^j_\alpha}{\partial v^i} V_j,$$

$$(1.25)\qquad S_{ij} = \frac{\partial V_i}{\partial v^j} - \frac{\partial V_j}{\partial v^i},$$

where $R^i_{\alpha\beta}$ is the curvature of the non-linear connection HE (see (1.42) of [1]). We call $R_{\alpha\beta}$, $P_{\alpha i}$ and S_{ij} the *horizontal, mixed* and respectively *vertical electromagnetic tensor fields.*

As Lagrangians for gauge fields H_α and V_i we may consider L_H', L_{HV}' and L_V' given by (2.68), (2.69) and respectively (2.70) from [2]. However, the pseudo-Riemannian metrics $\eta_{\alpha\beta}(x, v)$ and $g_{ij}(x, v)$ on vector bundles HE and respectively VE come from the exterior of the geometry of both M and E. That why, we consider the following particular Lagrangians

$$(1.26)\qquad L_H = -\tfrac{1}{4}\eta^{\alpha\beta}\eta^{\gamma\mu}R_{\alpha\gamma}R_{\beta\mu},$$

$$(1.27)\qquad L_{HV} = -\tfrac{1}{2}\eta^{\alpha\beta}g^{ij}(x)P_{\alpha i}P_{\beta j},$$

$$(1.28)\qquad L_V = -\tfrac{1}{4}g^{ij}(x)g^{hk}(x)S_{ih}S_{jk},$$

where $g^{ij}(x)$ and $\eta^{\alpha\beta}$ are the entries of the inverse matrix of a pseudo-Riemannian metric $g_{ij}(x)$ on the vector bundle E and respectively of the pseudo-Riemannian metric $\eta_{\alpha\beta}$ on M given by (1.1). Hence by the general theory in [2] we have the following three full Lagrangians :

$$\mathcal{L}_H(x, v) = \mathcal{L}_0(x, v) + \mathcal{L}_I(x, v) + L_H(x, v), \tag{1.29}$$

$$\mathcal{L}_{HV}(x, v) = \mathcal{L}_0(x, v) + \mathcal{L}_I(x, v) + L_{HV}(x, v), \tag{1.30}$$

$$\mathcal{L}_V(x, v) = \mathcal{L}_0(x, v) + \mathcal{L}_I(x, v) + L_V(x, v). \tag{1.31}$$

We are now concerned with equations of motion, currents and conservation laws. First we recall the equations of motion (1.22) from [3]

$$\frac{\partial \mathcal{L}}{\partial Q^A} - \frac{\delta}{\delta x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^\alpha}\right)}\right) - \frac{\partial}{\partial v^i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial v^i}\right)}\right) = E_A, \tag{1.32}$$

where in our case we have (see (2.27) and (2.28) of [1])

$$E_A = \left(\frac{1}{V}\frac{\partial V}{\partial x^\alpha} - \frac{\partial N^i_\alpha}{\partial v^i}\right)\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^\alpha}\right)}. \tag{1.33}$$

In (1.32) and (1.33) by $\mathcal{L}(x, v)$ we mean any of the three full Lagrangians $\mathcal{L}_H(x, v)$, $\mathcal{L}_{HV}(x, v)$ and $\mathcal{L}_V(x, v)$. By using (1.11), (1.22) and (1.26)–(1.31) we get

$$\frac{\partial \mathcal{L}}{\partial Q^2} = \frac{\partial \mathcal{L}}{\partial \bar{\Phi}} = i\eta^{\alpha\beta}\gamma_\alpha \frac{\delta \Phi}{\delta x^\beta} - m\Phi - g\eta^{\alpha\beta}\gamma_\alpha \Phi H_\beta, \tag{1.34}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^2}{\delta x^\alpha}\right)} = \frac{\partial \mathcal{L}}{\partial\left(\frac{\delta \bar{\Phi}}{\delta x^\alpha}\right)} = 0; \quad \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^2}{\partial v^i}\right)} = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \bar{\Phi}}{\partial v^i}\right)} = 0. \tag{1.35}$$

Hence the first equation of motion follows from (1.32) by using (1.33)–(1.35) :

$$i\eta^{\alpha\beta}\gamma_\alpha\left(\frac{\delta}{\delta x^\beta} + igH_\beta\right)\Phi = m\Phi, \tag{1.36}$$

which we call the *generalized Dirac–Maxwell equations* on a vector bundle. The second group of Maxwell equations on E follows from our Bianchi identities obtained in [3] for covariant derivatives of strength fields. More precisely, (2.15), (2.16), (2.21) and (2.23) become in our case

$$\sum_{(\alpha,\beta,\gamma)}^{\text{cyclic}} \{R_{\alpha\beta|\gamma} + R^i_{\alpha\beta} P_{\gamma i}\} = 0, \tag{1.37}$$

$$\sum_{(i,j,k)}^{\text{cyclic}} \{S_{ij||k}\} = 0, \tag{1.38}$$

$$P_{\alpha i||j} - P_{\alpha j||i} + S_{ij|\alpha} = 0, \tag{1.39}$$

and respectively

$$P_{\alpha i|\beta} - P_{\beta i|\alpha} - R_{\alpha\beta||i} = S_{ij} R^{j}_{\alpha\beta}. \tag{1.40}$$

According to (1.47) from [3] we obtain for Lagrangians $\mathcal{L}_H$, $\mathcal{L}_{HV}$ and $\mathcal{L}_V$ the following horizontal currents :

$$\overset{(h)}{J}{}^{\alpha}_{H} = \overset{(h)}{J}{}^{\alpha}_{HV} = \overset{(h)}{J}{}^{\alpha}_{V} = -g\eta^{\alpha\beta}\bar{\Phi}_{\gamma_\beta}\Phi. \tag{1.41}$$

On the other hand, by (1.48) of [3] we get the following vertical currents :

$$\overset{(v)}{J}{}^{i}_{H} = \overset{(v)}{J}{}^{i}_{HV} = \overset{(v)}{J}{}^{i}_{V} = 0. \tag{1.42}$$

Hence the conservation laws (1.49) of [3] become

$$\overset{(h)}{J}{}^{\alpha}_{H|\alpha} = \overset{(h)}{J}{}^{\alpha}_{HV|\alpha} = \overset{(h)}{J}{}^{\alpha}_{V|\alpha} = -g\eta^{\alpha\beta}\left(\frac{1}{V}\frac{\partial V}{\partial x^{\alpha}} - \frac{\partial N^{i}_{\alpha}}{\partial v^{i}}\right)\bar{\Phi}_{\gamma_\beta}. \tag{1.43}$$

Remark. It is esay to check that all equations of motion (1.36)—(1.40), the currents (1.41) and (1.42) and the conservation laws (1.43) have the same simple form in case $\eta_{\alpha\beta}$ and $g_{ij}(x)$ from (1.26)—(1.28) are replaced by arbitrary metrics $\eta_{\alpha\beta}(x, v)$ and $g_{ij}(x, v)$ on the vector bundles HE and VE respectively.

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TEORIE GAUGE PE UN FIBRAT VECTORIAL

IV. O aplicaţie în teoria cîmpului electromagnetic

(Rezumat)

În lucrările [1] ... [3] s-a dezvoltat un aparat matematic necesar unei teorii gauge pe un fibrat vectorial. În prezenta lucrare se aplică această teorie generală în cazul concret al teoriei cîmpului electromagnetic. Se obţin astfel ecuaţiile de mişcare, legile de conservare şi identităţile Bianchi pentru o teorie a cîmpului electromagnetic pe un fibrat vectorial.

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D.C. 517.1

THE DENSITY OF BOUNDED AND FINITE-DIRICHLET INTEGRAL HARMONIC FUNCTIONS ON STANDARD H-CONES (II)

BY

LILIANA POPA

In the case of an harmonic space, the set of all potentials can be characterized by the means of a Green function. All the super-harmonic functions which don't have an integral representation, relative to the Green kernel, are harmonic.

Les be $S=S^{**}$ a standard H-cone of functions on a semi-saturated set X[1], which satisfies the axioms A and D. We suppose that 1 is substractible [2]. If φ is an H-morphism for which S is autodual, we denote $[s, s]=\varphi(s)(s)$. In [5], we have defined the Green function $g(x, y)=\varphi^{-1}(x)(y)$ and we have obtained a consistent system of Green functions $(G_U)_{U\subset X}$, U naturally open.

Let be $S_\Delta=\{s\in S \mid [s, s]<\infty\}$. If P is the set of all I-potentials and S_P is the set of all nearly continuous elements, then from [1], we know that $S_\Delta\subset P\subset S_P$.

In [5], we have established that for any $p\in S_P$, there exists a unique non-negative measure such that

$$p(x)=\int_X g(x, y)\,d\sigma(p)(y).$$

For the bounded substractible elements h, we have $\sigma(h)=0$ [5]. This measure of representation leads us to the definition of a gradient measure denoted

$$\delta_{[f,g]}=\tfrac{1}{2}(f\sigma(g)+g\sigma(f)-\sigma(fg)),$$

for any $f, g\in[S_M]$, naturally continuous. S_M means the set of all bounded elements of S. In [4] we have shown that $[S_M]$ can be organized as an algebra with respect to the usual product of functions.

If we drop out the hypothesis of boundness, among the substractible elementscan be also founded some potentials. Therefore we have considered the set $\mathcal{R}=\{f: X\to\mathbb{R}_+ \mid \exists(U_i)_{i\in I}$, naturally open, $X=\bigcup_{i\in I} U_i f\mid_{U_i}\in[\varphi_M(U_i)]\}$ where $\varphi_M(U_i)$ is the set of all bounded elements of $\varphi(U)$ and $\varphi(U)$ is the localization of the H-cone S.

We have proved in [7], that if p, q, are potentials, h is substractible and $p, q, h\in\mathcal{R}$, then $\delta_{[p,h]}(1)=0$ and $\delta_{[p,q]}(1)=[p, q]$.

We denoted by $\mathcal{H}_D^+=\{u\in S_H\cap\mathcal{R},\ \delta_h(1)<\infty\}$.

In [5] we have established that for any generator q, such that $[q,1]<\infty$, the map

$$u \to \|u\| = \delta_u^{1/2}(1) + \left(\int_X u^2 d\sigma(q)\right)$$

defines a norm on $\mathcal{H}_D^+$.

From [7] we recall that for any bounded element of S, the next conditions are equivalent :

1. $u \in \mathcal{H}_{MD}^+$;
2. $u = \bigvee_{\alpha>0} (u \curlywedge \alpha)$ and $\sup_{\alpha>0} \delta_{u \wedge \alpha}(1) < \infty$.

We want to extend this result without the hypothesis of boundness. We suppose thas the next axiom is fulfilled :

There exists $(D_i)_{i \in I}$, naturally open, such that $\bigcup_{i \in I} D_i = X$ and for any $i \in I$, there exists $\alpha_i > 0$, for which

$$\sup_{X \in D_i} u(x) \leqslant \alpha_i \inf u(x), \quad \text{for any} \quad u \in \mathcal{H}_D^+.$$

In [5] we have shown the completness of $\mathcal{H}_D^+$, relative to the previous norm.

T h e o r e m. *$\mathcal{H}_{MD}^+$ is densly in $\mathcal{H}_D^+$.*

P r o o f. Let be $u \in \mathcal{H}_D^+$ and $v_\alpha = u \curlywedge \alpha$. There exists $v = \lim_{\alpha \to \infty} v_\alpha$, and $v \in S_H$, $v \leqslant u$. We want to prove that $v = u$. Because $u \in \mathcal{R}$, it results that u is continuous and $A_\alpha = [u \geqslant \alpha]$ is closed, Then we have

$$\max(u, \alpha) = \alpha \quad \text{on} \quad X \setminus A_\alpha,$$

$$\delta_{\max(u,\alpha)} = 0 \quad \text{on} \quad X \setminus A_\alpha.$$

Hence

$$\delta_{u - \min(u,\alpha)} \mid_{X \setminus A_\alpha} = \delta_{\max(u,\alpha)}(X \setminus A_\alpha) = 0.$$

Because

$$\delta_{\max(u,\alpha)} + \delta_{\min(u,\alpha)} = \delta_u + \delta_\alpha = \delta_u,$$

it results

$$\delta_{u - \min(u,\alpha)}(1) = \delta_{\max(u,\alpha)}(A_\alpha) \leqslant \delta_u(A_\alpha).$$

Passing to limit we get

$$\lim_{\alpha \to \infty} \delta_{u - \min(u,\alpha)}(1) \leqslant \lim_{\alpha \to \infty} \delta_u(A_\alpha) = 0. \tag{1}$$

For any $\alpha, \alpha' \in \mathbb{R}_+$, there exists $g_{\alpha\alpha'} \in [S_P]$, such that

$$u \wedge \alpha - u \wedge \alpha' = u \curlywedge \alpha - u \curlywedge \alpha' + g_{\alpha\alpha'}, \tag{2}$$

and using the previous results

$$\delta_{u \wedge \alpha - u \wedge \alpha'}(1) + \delta_{g_{\alpha\alpha'}}(1) = \delta_{u \wedge \alpha - u \wedge \alpha'}(1). \tag{3}$$

From (1) and the positivity of the gradient (4), we deduce

$$\delta_{u \wedge \alpha - u \wedge \alpha'}^{\frac{1}{2}}(1) \leqslant \delta_{u - \min(u,\alpha)}^{\frac{1}{2}}(1) + \delta_{u - \min(u,\alpha')}^{\frac{1}{2}}(1) \to 0 \qquad (\alpha, \alpha' \to \infty)$$

and from (3) we get

$$\delta^{\frac{1}{2}}_{u\curlywedge\alpha-u\curlywedge\alpha'}-(1)\to 0, \quad \text{for} \quad \alpha,\alpha'\to\infty. \tag{4}$$

Using the convergence theorem of Lebesgue, we obtain

$$\int_X (u\curlywedge\alpha-u\curlywedge\alpha')^2 d\sigma(q)\to 0, \qquad \alpha,\alpha'\to\infty. \tag{5}$$

From (5) and (4) it results that $(u\curlywedge\alpha)_{\alpha\in R}$ is a Cauchy sequence. There exists $v\in\mathcal{H}^+_D$ such that

$$\|v-u\curlywedge\alpha\|\to 0. \tag{6}$$

But for any $\alpha>0$, there exists $p_\alpha\in S_P\cap S_M$ such that

$$u\wedge\alpha=u\curlywedge\alpha+p_\alpha=\delta^{\frac{1}{2}}_{p_\alpha-u+v}(1)=\delta^{\frac{1}{2}}_{u\wedge\alpha-u\curlywedge\alpha-u+v}(1)\leqslant\delta^{\frac{1}{2}}_{u\wedge\alpha-u}(1)+\delta^{\frac{1}{2}}_{u\curlywedge\alpha-v}(1)\to 0;$$

$$\delta_{p_\alpha-u+v}(1)=\delta_p(1)+\delta_{u-v}(1).$$

From the previous calculus, we get

$$\lim_{\alpha\to\infty}\delta_{p_\alpha}(1)=0.$$

Let be $p\in S_0$; then

$$\int_X p_\alpha d\sigma(p)=\delta_{[p_\alpha,p]}(1)\leqslant\delta^{\frac{1}{2}}_{p_\alpha}(1)\delta^{\frac{1}{2}}_{p}(1)\to 0, \qquad \alpha\to\infty.$$

Hence (p_α) converges to 0 in the natural topology. There exists $x_0\in X$, such that $\lim\limits_{n\to\infty} p_{\alpha_n}(x_0)=0$. Hence $\lim\limits_{n\to\infty}(u\wedge\alpha_n)(x_0)=\lim\limits_{n\to\infty}(u\curlywedge\alpha_n)(x_0)$ and $u(x_0)=v(x_0)$. From $v\leqslant u$, we deduce $u-v\in S_H$. Because $(u-v)(x_0)=0$ the equality $u=v$ is valid.

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DENSITATEA FUNCŢIILOR ARMONICE MĂRGINITE, CU INTEGRALĂ DIRICHLET FINITĂ, ÎN *H*-CONURI STANDARD

(Rezumat)

În [5] s-a arătat că mulţimea elementelor substractibile cu integrală Dirichlet finită, pozitive, este completă în raport cu o normă introdusă cu ajutorul gradientului. Caracterizarea dată în [6] a elementelor substractibile mărginite, cu integrală Dirichlet finită, conduce la un rezultat de densitate a acestora, în prezenţa unei axiome de tip Harnack.

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THÉORÈMES DE TYPE NIEMYTZKI—EDELSTEIN DANS LES ESPACES T-MÉTRIQUES

PAR

TEMISTOCLE BÎRSAN

1. Introduction. Le but de cette Note est de donner une extension du théorème de point fixe dû à V. Niemytzki [6] et M. Edelstein [5] au cadre des espaces T-métriques.

Définition 1. Une fonction $d: X\times X\to \mathbb{R}_+$ est une *T-métrique* sur X si elle vérifie les conditions suivantes : I) $d(x, y)=0$ si et seulement si $x=y$: II) $d(x, y)=d(y, x)$ pour tous $x, y\in X$, et III) $d(x, z)\leqslant T(d(x, y), d(y, z))$ pour tous $x, y, z\in X$, $x\neq y\neq z\neq x$, où $T: \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ est une fonction quelconque (elle sera appelée *t-fonction*, c.-à-d. fonction triangulaire).

Le triplet (X, d, T) s'appelle *espace T-métrique*. Si la t-fonction T satisfait aux conditions à la limite

$$T(t, 0)=T(0, t)=t, \quad t\in\mathbb{R}_+, \tag{1}$$

alors, dans l'inégalité triangulaire III) les points x, y, z peuvent coïncider. On trouve certaines considérations topologiques sur les espaces T-métriques et quelques théorèmes de point fixe dans [1], [2], [3] et [4]. Rappellons un minimum de résultats dont nous aurons besoin.

Théorème A [2]. *Un espace T-métrique* (X, d, T) *dont la t-fonction T satisfait à la condition*

$$\lim_{a\to 0,\, b\to 0} T(a, b)=0 \tag{2}$$

est métrisable.

Théorème B [4]. *Soit* (X, d, T) *un espace T-métrique dont la t-fonction T vérifie les conditions suivantes :*

1° $T(0, b)\leqslant b$, *pour tout* $b\in\mathbb{R}_+$;

2° $t\to T(t, b)$ *est une application scs (c.-à-d. semi-continue supérieurement) au point* $t=0$ *uniformément par rapport à* $b\in K$, *quel que soit l'ensemble compact* $K\subset\mathbb{R}_+$.

Alors la T-métrique d est continue.

Désignons par $\mathcal{P}_{bf}(X)$ l'ensemble des parties de X non vides, bornées et fermées. On sous-entend le sens de $\mathcal{P}_b(X)$. Comme d'habitude, on introduit $H(.\,,.): \mathcal{P}_b(X)\times\mathcal{P}_b(X)\to\mathbb{R}_+$ par

$$H(A, B)=\max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}, \tag{3}$$

où $d(a, B)$ est la distance du point a à l'ensemble B.

Théorème C [3]. *Soit (X, d, T) un espace T-métrique dont T vérifie les conditions : 1° $T(t, 0) = T(0, t) = t$, pour tout $t \in \mathbb{R}_+$; 2° $T(a, b) \leqslant T(c, d)$ si $a \leqslant c$ et $b \leqslant d$; 3° T est scs sur $\mathbb{R}_+ \times \mathbb{R}_+$ par rapport à chaque variable. Alors H définie par (3) est une T-métrique sur $\mathcal{P}_{bf}(X)$.*

2. Théorèmes de point fixe. Soient X et Y deux espaces topologiques. On dit que $F : X \to \mathcal{P}(Y)$ ($Fx \neq \Phi$ pour tout $x \in X$) est une application multivoque scs si pour tout $x \in X$ et tout voisinage V de l'ensamble Fx il y a un voisinage U de x de sorte que $F(U) \subset V(F(U) = \cup\{Fx\ ;\ x \in U\})$.

Lemme [8, Th. 1 et Remarque 3]. *Si $F : X \to \mathcal{P}(Y)$ est une application multivoque scs tel que Fx est un ensemble non vide et compact quel que soit $x \in X$, et $h : X \times Y \to \mathbb{R}$ est une fonction sci (c.-à-d. semi-continue inférieurement), alors la fonction $m : X \to \mathbb{R}$ définie par $m(x) = \inf\{h(x, y)\ ;\ y \in Fx\}$ est sci sur X.*

Nous aurons en vue des applications multivoques $F : X \to \mathcal{P}_{bf}(X)$ qui satisfont à la condition

$$H(Fx, Fy) < \varphi(d(x, y), d(x, Fx), d(y, Fy)), \quad x \neq y, \tag{4}$$

où $\varphi : \mathbb{R}_+^3 \to \mathbb{R}_+$ a les propriétés suivantes : a) φ est croissante par rapport à chaque variable, et b) $\varphi(t, t, t) \leqslant t$, pour tout $t > 0$.

Le résultat suivant a une démonstration commune.

Théorème 1. *Soit (X, d, T) un espace T-métrique de sorte que*

1° $T(0, b) \leqslant b$, pour tout $b \in \mathbb{R}_+$, et

2° $t \to T(t, b)$ est scs au point $t = 0$ uniformément par rapport à $b \in K$, quel que soit l'ensemble compact $K \subset \mathbb{R}_+$.

Supposons que l'espace (X, d, T) est compact et $F : X \to \mathcal{P}_{bf}(X)$ est une application multivoque scs satisfaisant à la condition (4).

Alors F possède un point fixe.

Démonstration. Les hypothèses 1° et 2° entraînent la relation (2) et, grâce au Théorème A, l'espace donné est métrisable. D'autre part, le Théorème B garantit que la T-métrique d est une fonction continue sur $X \times X$. D'après le Lemme, la fonction $x \to d(x, Fx)$, $x \in X$, est sci. L'espace X étant compact, il existe un point $u \in X$ tel que

$$d(u, Fu) = \inf\{d(x, Fx) :\ x \in X\}. \tag{5}$$

Montrons que u est un point fixe de F. En effet, supposons que $u \notin Fu$. Alors, on a $d(u, Fu) > 0$. Par suite de la compacité de Fu et la continuité de d, il y a $z \in Fu$ ainsi que $d(u, z) = d(u, Fu)$. Il suit que $u \neq z$ et, par conséquent,

$$H(Fu, Fz) < \varphi(d(u, z),\ d(u, Fu),\ d(z, Fz)).$$

Vu la relation (5), on a $d(u, z) = d(u, Fu) \leqslant (z, Fz)$. En tenent compte des propriétés de φ, on obtient

$$H(Fu,\ Fz) < d(z, Fz). \tag{6}$$

Mais $z \in Fu$ implique que

$$d(z, Fz) \leqslant H(Fu, Fz),$$

ce qui est en contradiction avec (6). La démonstration est achevée.

On peut enlever l'hypothèse „F est scs" si on considère que F satisfait à la condition

(7) $$H(Fx, Fy) < a_1 d(x, y) + a_2 d(x, Fx) + a_3 d(y, Fy), \qquad x \neq y,$$

où $a_1, a_2, a_3 \in \mathbb{R}_+$ et $a_1 + a_2 + a_3 = \ .$

Théorème 2. *Si les hypothèses du Théorème 1 relativement à l'espace (X, d, T) sont satisfaites et $F : X \to \mathcal{P}_{bf}(X)$ est une application multivoque qui vérifie la condition (7), alors elle a au moins un point fixe.*

Démonstration. On a déjà vu que l'espace X est métrisable et la T-métrique d est continue. Soit $m = \inf\{d(x, Fx)\ ;\ x \in X\}$. Il y a une suite $(x_n)_n$ de sorte que $d(x_n, Fx_n) \underset{n}{\to} m$. Puisque d est continue et Fx_n est compact, on peut choisir $y_n \in Fx_n$ ainsi que $d(x_n, y_n) = d(x_n, Fx_n)$, pour tout $n \in \mathbb{N}$. L'espace X étant compact, il existe une sous-suite $(y_{n_k})_k$ de la suite $(y_n)_n$ telle que $y_{n_k} \underset{k}{\to} u$. Soient $\alpha = \max\ \{1, \sup\limits_n d(x_n, y_n), d(u, Fu)\}$ et $\varepsilon \in (0, 1)$. Vu les hypothèses 1° et 2° il y a $\delta > 0$ tel que

(8) $$T(t, b) < b + \varepsilon, \quad t < \delta \quad \text{et} \quad b \in [0, 2\alpha].$$

En vertu de la condition (7), on peut écrire

(9) $$H(Fu, Fx_{n_k}) < a_1 d(u, x_{n_k}) + a_2 d(u, Fu) + a_3 d(x_{n_k}, Fx_{n_k})$$

pour $k \geqslant k_0$ (car le cas où l'égalité $x_{n_k} = u$ a lieu pour une infinité d'indices k est trivial). En utilisant l'inégalité triangulaire III), on obtient

(10) $$H(Fu, Fx_{n_k}) < a_1 T(d(u, y_{n_k}), d(y_{n_k}, x_{n_k})) + a_2 d(u, Fu) + a_3 d(x_{n_k}, y_{n_k})$$

(le cas où $y_{n_k} = x_{n_k}$ est trivial, tandis que si $y_{n_k} = u$ on passe directement de (9) à (12)). Parce que $(y_{n_k})_k$ converge vers u, on a

(11) $$d(u, y_{n_k}) < \delta \quad \text{dès que} \quad k \geqslant k_0$$

(on peut considérer le même rang k_0 comme ci-dessus). De (8), (11), et $d(x_{n_k}, y_{n_k}) \leqslant \alpha$, $k \in \mathbb{N}$, il suit

$$T(d(u, y_{n_k}), d(y_{n_k}, x_{n_k})) < d(y_{n_k}, x_{n_k}) + \varepsilon, \quad k \geqslant k_0.$$

D'ici et de (10), il résulte

(12) $$H(Fu, Fx_{n_k}) < a_1 \varepsilon + (a_1 + a_3) d(x_{n_k}, y_{n_k}) + a_2 d(u, Fu), \quad k \geqslant k_0,$$

donc

$$H(Fu, Fx_{n_k}) < a_1 \varepsilon + (a_1 + a_2 + a_3)\alpha,$$

ou bien

(13) $$H(Fu, Fx_{n_k}) < 2\alpha, \quad k \geqslant k_0.$$

D'autre part, il existe $z_k \in Fu$ tel que

(14) $$d(z_k, y_{n_k}) \leqslant H(Fu, Fx_{n_k}),$$

en vertu de la continuité de d et la compacité de Fu. Alors, on a

$$d(u, Fu) \leqslant d(u, z_k) \leqslant T(d(u, y_{n_k}), d(y_{n_k}, z_k))$$

(on met au point facilement les cas où l'inégalité triangulaire ne peut pas être appliquée). En tenant compte de (8), (11), (13), et (14), on a

$$d(u, Fu) < H(Fu, Fx_{n_k}) + \varepsilon, \quad k \geqslant k_0. \tag{15}$$

Les relations (12) et (15) nous conduisent à

$$d(u, Fu) < (1 + a_1)\varepsilon + (a_1 + a_3) d(x_{n_k}, y_{n_k}) + a_2 d(u, Fu), \quad k \geqslant k_0,$$

d'où

$$(1 - a_2) d(u, Fu) < (1 + a_1)\varepsilon + (a_1 + a_3) d(x_{n_k}, y_{n_k})$$

ou, en passant à la limite pour $k \to \infty$,

$$(1 - a_2) d(u, Fu) \leqslant (1 + a_1)\varepsilon + (a_1 + a_3) m.$$

D'ici et du fait qu $\varepsilon > 0$ est arbitraire, on obtient que $d(u, Fu) \leqslant m$ et, donc,

$$d(u, Fu) = m = \inf \{d(x, Fx) \,;\, x \in X\}. \tag{16}$$

La relation (16) ayant le rôle joué par (5) dans le Théorème 1, on démontre que u est un point fixe de F comme dans ce théorème.

Les hypothèses du Théorème 2 n'entraînent pas que F est scs, donc ce théorème n'est pas une conséquence du Théorème 1.

Exemple. Soient $X = \{z \,;\, |z| \leqslant 1, \mathcal{R}e\, z \geqslant 0, \mathcal{I}m\, z \geqslant 0\}$, d la distance euclidienne, et $T(a, b) = a + b$. Alors (X, d, T) est un espace T-métrique compact. Soit $F: X \to \mathcal{P}_{bf}(X)$ définie par

$$Fz = \begin{cases} \{0\}, & |z| = 1 \\ \{t \,;\, |t| \leqslant \frac{1}{3} |z|, \arg t = \arg z\}, & z \in X \text{ et } |z| < 1. \end{cases}$$

L'application multivoque F vérifie la condition (7) pour $a_1 = a_2 = a_3 = \frac{1}{3}$. De même, elle a $z = 0$ pour le seul point fixe. Mais F n'est pas scs (l'image inverse par F de l'ensemble fermé $\{z \,;\, |z| = \frac{1}{6}, \mathcal{R}e\, z \geqslant 0, \mathcal{I}m\, z \geqslant 0\}$ est $\{z \,;\, \frac{1}{2} \leqslant |z| < 1, \mathcal{R}e\, z \geqslant 0, \mathcal{I}m\, z \geqslant 0\}$ qui n'est pas également fermé).

Corollaire. *Soit (X, d, T) un espace T-métrique compact dont la t-fonction T vérifie les hypothèses 1° et 2° du Théorème 1. Si $F: X \to \mathcal{P}_{bf}(X)$ est une application multivoque contractive, c.-à-d. a lieu*

$$H(Fx, Fy) < d(x, y), \quad x \neq y, \tag{17}$$

alors elle a au moins un point fixe.

Remarque 1. Lorsque $T(a, b) = a + b$, l'espace T-métrique (X, d, T) devient un espace métrique (classique) et le Corollaire n'est autre chose que le théorème de Niemytzki-Edelstein pour les applications multivoques [7, Corollaire 1.2].

Remarque 2. Dans le cas particulier où F est une fonction (univoque), on montre immédiatement l'unicité du point fixe dont l'existence est garantie par les résultats ci-dessus.

Remarque 3. Dans les considérations précédentes H n'a pas été généralement une T-métrique sur $\mathcal{P}_{bf}(X)$. Le Théorème C indique des conditions suffisantes pour que H le soit.

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TEOREME DE TIP NEMÎŢKI-EDELSTEIN ÎN SPAŢII *T*-METRICE

(Rezumat)

Se dau cîteva extensiuni ale unei teoreme de punct fix datorită lui V. Nemîţki [6] şi M. Edelstein [5] la multifuncţii pe spaţii *T*-metrice.

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OBSERVATIONS SUR DES PAIRES D'APPLICATIONS MULTIVOQUES D'UN CERTAIN TYPE DE CONTRACTIVITÉ

PAR

NICOLETA NEGOESCU

Dans la Note présente nous nous proposons de démontrer quelques théorèmes de points fixes communs pour des paires d'applications multivoques définies sur des espaces métriques compacts ou sur des espaces métriques complets qui satisfont aux inégalités contractives (1) ou (2) ou (3), en partant des résultats ressemblants obtenus par B. Fisher [2], B. E. Rhoades [6], K. L. Singh et J. H. M. Whitfield [8], T. Kubiak [4], H. Kaneko [5], I. A. Rus [7] pour d'autres types de conditions contractives.

Soient (X, d) un espace métrique et A, B deux ensembles nonvides de X. Alors on définit :

$$D(A, B) = \inf \{d(a, b) : a \in A,\ b \in B\},$$

$$\delta(A, B) = \sup \{d(a, b) : a \in A,\ b \in B\}$$

(où A ou B peut être un ensemble formé d'un seul point),

$$H(A, B) = \max \{\sup_{a \in A} D(a, B),\ \sup_{b \in B} D(b, A)\},$$

$$BN(X) = \{A : A \subset X,\ A \neq \emptyset,\ A \text{ borné}\},$$

$$CL(X) = \{A : A \subset X,\ A \neq \emptyset,\ A \text{ fermé}\},\quad CB(X) = BN(X) \cap CL(X)$$

et

$$Cpt(X) = \{A : A \subset X,\ A \neq \emptyset,\ A \text{ compact}\}.$$

Observations. La fonction D est continue (voir [3]).

Évidemment on a : $D(x, A) \leqslant \delta(x, A)$ et $\delta(A, B) \geqslant H(A, B)$.

La fonction H est une métrique sur $CB(X)$ (es sur $Cpt(X)$) appellée la métrique de Hausdorff (voir [1]).

Théorème 1. *Soient (X, d) un espace métrique compact et $S, T : X \to CL(X)$ deux applications multivoques. Soint S ou T continue.*

Si on suppose que pour tous $x, y \in X$, $x \neq y$, on a

$$(1)\qquad H^2(Sx, Ty) < \max \{d^2(x, y), D(x, Sx)D(y, Ty), D(x, Ty)D(y, Sx)\},$$

alors S ou T a un point fixe.

Démonstration. On peut supposer que S est un application multivoque continue et on définit sur X une fonction réelle f, telle que $f(x) =$

$=D(x, Sx)$. Car f est une fonction continue sur X ([3]), il suit que f atteint son minimum. Soit $f(x_0)=\min\{f(x) : x\in X\}$.

On choisit $x_1\in Sx_0$, $x_2\in Tx_1$ et $x_3\in Sx_2$ tels que $d(x_0, x_1)=D(x_0, Sx_0)$, $d(x_1, x_2)=D(x_1, Tx_1)$ et $d(x_2, x_3)=D(x_2, Sx_2)$.

Nous montrerons maintenant qu'on a ou $D(x_0, Sx_0)=0$ ou $D(x_1, Tx_1)=0$, c'est-à-dire ou $x_0\in Sx_0$ ou $x_1\in Tx_1$.

Nous supposons que $D(x_0, Sx_0)>0$ et $D(x_1, Tx_1)>0$. Alors il suit

$$d^2(x_1, x_2)\leqslant H^2(Sx_0, Tx_1)<\max\{d^2(x_0, x_1), D(x_0, Sx_0)D(x_1, Tx_1), D(x_0, Tx_1)D(x_1, Sx_0)\}=\max\{d^2(x_0, x_1), d(x_1, x_2)d(x_0, x_1)\},$$

car $x_1\in Sx_0$ et donc $D(x_1, Sx_0)=0$.

Nous avons deux cas :

a) $\max\{d^2(x_0, x_1), d(x_1, x_2)d(x_0, x_1)\}=d^2(x_0, x_1)$; alors $d(x_1, x_2)<d(x_0, x_1)$;

b) $\max\{d^2(x_0, x_1), d(x_1, x_2)d(x_0, x_1)\}=d(x_1, x_2)d(x_0, x_1)$ et donc $d^2(x_1, x_2)<d(x_1, x_2)d(x_0, x_1)$.

Mais $d(x_1, x_2)=D(x_1, Tx_1)>0$ et il suit que $d(x_1, x_2)<d(x_0, x_1)$.

De la même manière

$$d^2(x_2, x_3)\leqslant H^2(Sx_2, Tx_1)<$$

$$<\max\{d^2(x_1, x_2), D(x_2, Sx_2)D(x_1, Tx_1), D(x_1, Sx_2)D(x_2, Tx_1)\}$$

et il suit $d^2(x_2, x_3)<d^2(x_1, x_2)$ ou $d^2(x_2, x_3)<d(x_2, x_3)d(x_1, x_2)$. Dans le deuxième cas $d(x_2, x_3)=0$ conduit à une contradiction. Donc dans ces deux cas on a $d(x_2, x_3)<d(x_1, x_2)$.

Donc il suit $D(x_2, Sx_2)=d(x_2, x_3)<d(x_1, x_2)<d(x_0, x_1)=f(x_0)$, ce que contredit la minimalité de $f(x_0)$. Donc $D(x_0, Sx_0)=0$ ou $D(x_1, Tx_1)=0$ et $x_0\in Sx_0$ ou $x_1\in Tx_1$ et donc S ou T a un point fixe.

Observations. Le Théorème 1 est vrai aussi si $S=T$. Le Théorème 1 a lieu aussi pour $S, T : X\to X$, deux opérateurs.

Aussi nous avons la suivante

Conséquence 1. *Soient (X, d) un espace métrique compact, $T : X\to X$ un opérateur continu qui pour tous $x, y\in X$, $x\neq y$, satisfait a l'inégalité*

$$(1')\qquad d^2(Tx, Ty)<\max\{d^2(x, y), d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}.$$

Alors l'opérateur T a un point fixe $u\in X$.

Nous pouvons illustrer cette conséquence par le suivant

Exemple. Soit l'opérateur $T:[0, A]\to[0, A^2/(A+1)]$, $0<A<+\infty$, défini par $T(x)=x^2/(x+1)$. En considérant la distance $d(x, y)=|x-y|$, $x, y\in[1, A]$, on observe que sont satisfaites toutes les conditions de la Conséquence 1 et donc l'opérateur T a un point fixe $u\in[0, A]$.

Théorème 2. *Soient (X, d) un espace métrique compact et les applications multivoques $S, T : X\to CL(X)$ où S ou T est continue. Si on suppose que S et T satisfont à l'inégalité*

$$(2)\qquad \delta^2(Sx, Ty)<\max\{d^2(x, y), H(x, Sx)H(y, Ty), D(x, Ty)D(y, Sx)\}$$

pour tous $x, y\in X$, $x\neq y$, alors S ou T a un point fixe $u\in X$ tel que $Su=\{u\}$ ou $Tu=\{u\}$.

D é m o n s t r a t i o n. Soit S continue. On définit sur X une fonction réelle continue f, $f(x)=H(x, Sx)$ et donc f atteint son minimum dans un point $x_0 \in X$. Soit $f(x_0)=\min\{f(x), x \in X\}$. On choisit $x_1 \in Sx_0$, $x_2 \in Tx_1$ et $x_3 \in Sx_2$ tels que $d(x_0, x_1)=H(x_0, Sx_0)$, $d(x_1, x_2)=H(x_1, Tx_1)$, $d(x_2, x_3)=H(x_2, Sx_2)$.

Nous supposons que $H(x_0, Sx_0)>0$ et $H(x_1, Tx_1)>0$, Alors, en procédant comme dans la démonstration du Théorème 1, on obtient que $d(x_2, x_3)<d(x_1, x_2)<d(x_0, x_1)$ ou $H(x_2, Sx_2)<H(x_1, Tx_1)<H(x_0, Sx_1)$, ce que contredit la minimalité de $f(x_0)=H(x_0, Sx_0)$. Donc, ou $H(x_0, Sx_0)=0$ ou $H(x_1, Tx_1)=0$, c'est-à-dire $Sx_0=\{x_0\}$ ou $Tx_1=\{x_1\}$.

Observation. Le Théorème 2 est vrai aussi pour une seule application multivoque $S=T: X \to CL(X)$.

T h é o r è m e 3. *Soient (X, d) un espace métrique complet et S, T deux applications multivoques de X dans $CB(X)$. Supposons qu'il existe une constante h, $0 \leqslant h<1$, telle que pour tous $x, y \in X$ on a*

$$H^2(Sx, Ty) \leqslant h^2 \max\{d^2(x, y), D(x, Sx)D(y, Ty), D(x, Ty)D(y, Sx)\}. \tag{3}$$

Alors les applications multivoques S et T ont un point fixe comun, c'est-à-dire il existe un point $x \in X$ tel que $x \in Sx \cap Tx$.

D é m o n s t r a t i o n. Nous supposons d'abord que $h=0$. Soient $x_0 \in X$, $x_1 \in Sx_0$ et $x_2 \in Tx_1$. Alors $D(x_1, Tx_1) \leqslant H(Sx_0, Tx_1)=0$ et donc $x_1 \in Tx_1$.

Aussi $D(x_1, Sx_1) \leqslant H(Tx_1, Sx_1)=0$ et il suit que $x_1 \in Sx_1 \cap Tx_1$.

Soient donc $h \neq 0$ et $x_0 \in X$, $x_1 \in Sx_0$, $x_2 \in Tx_1$. Nous construisons la suite (x_n) où $x_{2n-1} \in Sx_{2n-2}$, $x_{2n} \in Tx_{2n-1}$ sont tels que

$$d(x_{2n-1}, x_{2n}) \leqslant \frac{1}{\sqrt{h}} H(Sx_{2n-2}, Tx_{2n-1}),$$

$$d(x_{2n}, x_{2n+1}) \leqslant \frac{1}{\sqrt{h}} H(Sx_{2n}, Tx_{2n-1}).$$

On suppose d'abord qu'il existe $n \in \mathbb{N}$ tel que $x_n=x_{n+1}$.

Si n est pair nous avons $x_{2n} \in Sx_{2n}$ et il suit

$$D^2(x_{2n}, Tx_{2n}) \leqslant H^2(Sx_{2n}, Tx_{2n}) \leqslant$$

$$\leqslant h^2 \max\{d^2(x_{2n}, x_{2n}), D(x_{2n}, Sx_{2n})D(x_{2n}, Tx_{2n}), D(x_{2n}, Sx_{2n})D(x_{2n}, Tx_{2n})\}=0$$

car $d(x_{2n}, x_{2n})=0$ et $D(x_{2n}, Sx_{2n})=0$. Donc $x_{2n} \in Tx_{2n}$. On obtient le même résultat si n est impair.

Dans le cas $x_n \neq x_{n+1}$, pour chaque $n \in \mathbb{N}$, nous montrerons que la suite (x_n) est une suite de Cauchy. Il suit que

$$d^2(x_{2n}, x_{2n+1}) \leqslant \left(\frac{1}{\sqrt{h}}\right)^2 H^2(Sx_{2n}, Tx_{2n-1}) \leqslant \frac{1}{h} h^2 \max\{d^2(x_{2n-1}, x_{2n}),$$

$$D(x_{2n}, Sx_{2n})D(x_{2n-1}, Tx_{2n-1}), D(x_{2n}, Tx_{2n-1})D(x_{2n-1}, Sx_{2n})\},$$

donc nous avons obtenu que

$$d^2(x_{2n}, x_{2n+1}) \leqslant h \max\{d^2(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})d(x_{2n-1}, x_{2n})\}.$$

Il suit une des deux possibilités :

a) ou $\max\{d^2(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\, d(x_{2n-1}, x_{2n})\} = d^2(x_{2n-1}, x_{2n})$ et alors

$$d^2(x_{2n}, x_{2n+1}) \leqslant h d^2(x_{2n-1}, x_{2n}) \text{ et } d(x_{2n}, x_{2n+1}) \leqslant \sqrt{h}\, d(x_{2n-1}, x_{2n});$$

b) ou $\max\{d^2(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}) d(x_{2n-1}, x_{2n})\} = d(x_{2n}, x_{2n+1}) d(x_{2n-1}, x_{2n})$ et alors

$$d^2(x_{2n}, x_{2n+1}) \leqslant h d(x_{2n}, x_{2n+1}) d(x_{2n-1}, x_{2n}) \text{ ou } d(x_{2n}, x_{2n+1}) \leqslant h d(x_{2n-1}, x_{2n})$$

(car $x_{2n} \neq x_{2n+1}$ et $d(x_{2n}, x_{2n+1}) > 0$).

Mais $\max\{\sqrt{h}, h : 0 \leqslant h < 1\} = \sqrt{h}$ et donc nous avons montré que

$$d(x_{2n}, x_{2n+1}) \leqslant \sqrt{h}\, d(x_{2n-1}, x_{2n}), \quad \forall n \in \mathbb{N}. \tag{4}$$

De la même manière on peut montrer que

$$d(x_{2n-1}, x_{2n}) \leqslant \sqrt{h}\, d(x_{2n-2}, x_{2n-1}). \tag{4'}$$

En répétant le raisonnement qui nous a conduit aux inégalités (4) et (4′) nous obtenons

$$d(x_{2n}, x_{2n+1}) \leqslant h^n d(x_0, x_1) \text{ et } d(x_{2n+1}, x_{2n+2}) \leqslant h^n d(x_1, x_2).$$

En notant par $r_0 = \max\{d(x_0, x_1), d(x_1, x_2)\}$, nous avons pour $m > n$

$$d(x_n, x_m) \leqslant \sum_{i=0}^{m-n-1} d(x_{n+1}, x_{n+i+1}) \leqslant \sum_{i=0}^{m-n-1} h^{n+i} r_0 \leqslant h^n r_0 (1-h)^{-1}$$

et $d(x_n, x_m) \to 0$ pour $n \to \infty$. Donc (x_n) est une suite de Cauchy. (X, d) étant un espace métrique complet, il suit que (x_n) est une suite convergente à un point $x \in X$ ($\lim\limits_{n\to\infty} x_n = x \in X$). Alors

$$D^2(x_{2n-2}, Tx) \leqslant H^2(Sx_{2n-2}, Tx) \leqslant$$

$$\leqslant h^2 \max\{d^2(x_{2n-2}, x), D(x_{2n-2}, Sx_{2n-2}) D(x, Tx), D(x_{2n-2}, Tx) D(x, Sx_{2n-2})\}$$

ou

$$D^2(x_{2n-2}, Tx) \leqslant$$

$$\leqslant h^2 \max\{d^2(x_{2n-2}, x), d(x_{2n-2}, x_{2n-1})) D(x, Tx), D(x_{2n-2}, Tx)\, D(x, x_{2n-1})\}.$$

Pour $n \to \infty$ nous avons

$$D^2(x, Tx) \leqslant h^2 \max\{d^2(x, x), d(x, x) D(x, Tx), D(x, Tx) d(x, x)\} = 0$$

ce que implique $D(x, Tx) = 0$ et donc $x \in Tx$.

De la même manière nous obtenons que $x \in Sx$ et la démonstration est finie.

Observations. Le Théorème 3 est vrai aussi pour $S = T : X \to CB(X)$.

Du Théorème 3 nous obtenons un résultat analogue pour deux opérateurs $S, T : X \to X$.

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OBSERVAŢII ASUPRA UNOR PERECHI DE MULTIFUNCŢII DE UN ANUMIT TIP DE CONTRACTIVITATE

(Rezumat)

Se demonstrează cîteva teoreme de punct fix comun pentru perechi de multifuncţii definite pe spaţii metrice compacte şi spaţii metrice complete care satisfac inegalităţi de tip (1) sau (2) sau (3).

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THE EQUATION $y''+Q(x)y=f(x)$ WITH „SMALL" Q AND f

BY

ADRIAN CORDUNEANU

We consider the linear differential equation

$$y''+Q(x)y=f(x), \qquad x\geqslant 0 \tag{1}$$

with Q and f continuous for $x\geqslant 0$ and we wish to prove that there exists a one-to-one correspondence between the set of its solutions and the set of the solutions of the simplest differential equation of the second order, namely $y''=0$, if the functions Q and f are "smal" in the sense which will be precised in what follows. More precisely, under reasonable conditions imposed to the functions Q and f, it will be proved that the graph of each solution $y=y(x)$ of (1) has asymptote of the equation $y=\alpha x+\beta$ for $x\to\infty$ and the cited here correspondence is given by $y=y(x)\leftrightarrow y=\alpha x+\beta$; α and β may take any real values.

L e m m a 1. *Assume that*

$$\int_0^\infty x\,|\,Q(x)\,|\,\mathrm{d}x<\infty, \qquad \int_0^\infty |\,f(x)\,|\,\mathrm{d}x<\infty. \tag{2}$$

Then, every solution $y=y(x)$ of the Eq. (1) *satisfies a majoration of the form*

$$|\,y(x)\,|\leqslant K_0+K_1x, \qquad x\geqslant 0, \tag{3}$$

where K_0 and K_1 are positive constants.

P r o o f. Let $y=y(x)$ be a solution of (1)—(2) ; we have

$$y(x)=y(1)+(x-1)y'(1)+\int_1^x (x-t)f(t)\mathrm{d}t-\int_1^x (x-t)Q(t)y(t)\mathrm{d}t, \qquad x\geqslant 1 \tag{4}$$

from which it follows

$$\frac{|\,y(x)\,|}{x}\leqslant\frac{|\,y(1)\,|}{x}+|\,y'(1)\,|+\int_1^x |\,f(t)\,|\,\mathrm{d}t+\int_1^x |\,Q(t)\,|\,|\,y(t)\,|\,\mathrm{d}t, \quad x\geqslant 1. \tag{5}$$

Denoting $u(x) = |y(x)|/x$, we have

$$u(x) \leqslant K_2 + \int_1^x t\,|Q(t)|\,u(t)\,dt, \qquad x \geqslant 1 \tag{6}$$

where $K_2 > 0$ is a constant. Now, the Gronwal-Bellman lemma implies

$$u(x) \leqslant K_2 \exp\left(\int_1^\infty t\,|Q(t)|\,dt\right) = K_1, \qquad x \geqslant 1 \tag{7}$$

which concludes the proof. Of course, the constants K_0 and K_1 depend of the given solution $y = y(x)$.

Lemma 2. *For each solution $y = y(x)$ of the problem (1)–(2), there exists*

$$\lim_{x\to\infty} y'(x) = \textit{finite}. \tag{8}$$

Proof. Integrating the Eq. (1) on the interval $[x_1, x_2]$, we obtain

$$\begin{aligned} |y'(x_2) - y'(x_1)| &\leqslant \left|\int_{x_1}^{x_2} f(x)\,dx\right| + \left|\int_{x_1}^{x_2} xQ(x)\frac{y(x)}{x}\,dx\right| \leqslant \\ &\leqslant \left|\int_{x_1}^{x_2} |f(x)|\,dx\right| + K\left|\int_{x_1}^{x_2} x\,|Q(x)|\,dx\right|, \end{aligned} \tag{9}$$

where $K > 0$ is a constant. This implies that the relation (8) is true. Of course, if $y = y(x)$ is a bounded solution, then we have $y'(x) \to 0$ when $x \to \infty$.

In view to construct bounded solutions for the Eq. (1), we consider the conditions

$$\int_0^\infty x\,|Q(x)|\,dx < \infty, \qquad \int_0^\infty x\,|f(x)|\,dx < \infty. \tag{10}$$

Lemma 3. *Under the hypothesis (10), for every real a, there exists a unique solution $y = y(x)$ of the Eq. (1), such that*

$$y(x) \to a, \qquad y'(x) \to 0 \quad \textit{when} \quad x \to \infty. \tag{11}$$

Proof. If $y = y(x)$ is a solution of (1) satisfying (11), we can write

$$y(x) = a + \int_x^\infty (t - x) f(t)\,dt - \int_x^\infty (t - x) Q(t) y(t)\,dt, \qquad x \geqslant 0. \tag{12}$$

Converisely, if the bounded continuous function $y = y(x)$ satisfies (12), then there exsts $y''(x)$, the Eq. (1) is verified for $x \geqslant 0$ and $y(x) \to a$, $y'(x) \to 0$

when $x \to \infty$. Because we know ([3], p. 450) that there exists a unique solution $y_1(x)$ of the homogeneous equation $y'' + Q(x)y = 0$, satisfying the conditions (11), it is not restrictively to consider here the case $a = 0$ only. It may be easily shown that this solution $y_1(x)$ may be obtained using the method of succesive approximations for the integral equation

$$y(x) = a - \int_x^\infty (t-x)Q(t)y(t)\,dt, \qquad x \geqslant 0. \tag{13}$$

Thus we consider the equation

$$y(x) = \varphi(x) - \int_x^\infty (t-x)Q(t)y(t)\,dt, \qquad x \geqslant 0, \tag{14}$$

where $\varphi = \varphi(x)$ is the bounded continuous function given by

$$\varphi(x) = \int_x^\infty (t-x)f(t)\,dt, \qquad x \geqslant 0 \tag{15}$$

and we put

$$y_0(x) = \varphi(x), \qquad y_{n+1}(x) = \varphi(x) - \int_x^\infty (t-x)Q(t)y_n(t)\,dt \tag{16}$$

for $x \geqslant 0$ and n = natural. We have

$$|\, y_1(x) - y_0(x) \,| \leqslant M \int_x^\infty t \,|\, Q(t) \,|\, dt = MR(x), \qquad x \geqslant 0, \tag{17}$$

where $M > 0$ is a constant, such that $|\, \varphi(x) \,| \leqslant M$ for $x \geqslant 0$. Assuming that

$$|\, y_n(x) - y_{n-1}(x) \,| \leqslant \frac{MR^n(x)}{n!}, \qquad x \geqslant 0 \tag{18}$$

we obtain

$$\begin{aligned} |\, y_{n+1}(x) - y_n(x) \,| &\leqslant \int_x^\infty t \,|\, Q(t) \,|\, \frac{MR^n(t)}{n!}\,dt = -\frac{M}{n!}\int_x^\infty R^n(t)R'(t)\,dt = \\ &= \frac{MR^{n+1}(x)}{(n+1)!}, \qquad x \geqslant 0, \end{aligned} \tag{19}$$

i.e. (18) holds for every natural n. Hence the series

$$y_0(x) + \sum_{n=1}^\infty (y_n(x) - y_{n-1}(x)), \qquad x \geqslant 0 \tag{20}$$

converges uniformly for $x \geqslant 0$ to a bounded continuous function $y = y(x)$. In other words, the sequence $\{y_n(x)\}$ converges uniformly on $[0, \infty)$ to $y(x)$ and making $n \to \infty$ in the second relation (16), we get that $y = y(x)$ is a solution of (1), satisfying the conditions $y(x) \to 0$, $y'(x) \to 0$ when $x \to \infty$. To prove its unicity, it suffices to show that the integral equation

$$y(x) = -\int_x^\infty (t-x) Q(t) y(t) \mathrm{d}t, \quad x \geqslant 0 \tag{21}$$

where $y = y(x)$ is a bounded continuous function, has the solution $y(x) \equiv 0$ only. Indeed, if we suppose $|y(x)| \leqslant N$ for $x \geqslant 0$, we have from (21) that $|y(x)| \leqslant NR(x)$ for $x \geqslant 0$ and integrating this inequality, we obtain

$$|y(x)| \leqslant N \frac{R^n(x)}{n!} \leqslant N \frac{R^n(0)}{n!}, \quad x \geqslant 0, \tag{22}$$

which implies $y(x) \equiv 0$.

L e m m a 4. *Consider the Eq.* (1) *under the hypothesis* (10). *Then it follows :*

i) *The set of all bounded solutions is characterized by*

$$\lim_{x \to \infty} y(x) = \textit{finite}, \quad \lim_{x \to \infty} y'(x) = 0. \tag{23}$$

ii) *The set of all unbounded solutions is characterized by*

$$\lim_{x \to \infty} y(x) = \textit{infinite}, \quad \lim_{x \to \infty} y'(x) = \textit{finite} \neq 0. \tag{24}$$

P r o o f. If suppose that $y = y(x)$ is a bounded solution and take into account the Lemma 2, it follows that the second relation (23) is true. Integrating the Eq. (1) on $[x, x_2]$ and making $x_2 \to \infty$, we obtain

$$y'(x) = -\int_x^\infty f(t) \mathrm{d}t + \int_x^\infty Q(t) y(t) \mathrm{d}t, \quad x \geqslant 0. \tag{25}$$

A new integration on the interval $[x_1, x_2]$ implies for $x_1 < x_2$

$$\begin{aligned} |y(x_2) - y(x_1)| &\leqslant \int_{x_1}^{x_2} \mathrm{d}x \int_x^\infty |f(t)| \mathrm{d}t + \int_{x_1}^{x_2} \mathrm{d}x \int_x^\infty |Q(t)| |y(t)| \mathrm{d}t \leqslant \\ &\leqslant \int_{x_1}^\infty t\, |f(t)| \mathrm{d}t + K_0 \int_{x_1}^\infty t\, |Q(t)| \mathrm{d}t, \end{aligned} \tag{26}$$

where $K_0 > 0$ is a constant. This inequality, together with the conditions (10), implies that there exists finite limit for $x \to \infty$ of the given bounded solution $y = y(x)$. Hence the part i) was proved. To justify the part ii), we shall write

the general solution of the Eq. (1) under the form

$$y(x)=\left(c_1-\int_0^x \frac{y_2(t)f(t)}{W_0}\,dt\right)y_1(x)+\left(c_2+\int_0^x \frac{y_1(t)f(t)}{W_0}\,dt\right)y_2(x), \tag{27}$$

where c_1 and c_2 are arbitrary real constants and $\{y_1(x),\ y_2(x)\}$ is a fundamental system of solutions of the equation $y''+Q(x)y=0$ ([3], p. 450), such that $y_1(x)\to 1$ and $y_1'(x)\to 0$, $y_2(x)/x\to 1$ and $y_2'(x)\to 1$ when $x\to\infty$. The constant $W_0\neq 0$ is the wronskian of the solutions $y_1(x)$ and $y_2(x)$.

The set of all unbounded solutions of the Eq. (1) —(10) is given by (27), where $c_2\neq c_2^*$, the constant c_2^* being defined as follows

$$c_2^*=-\int_0^\infty \frac{y_1(t)f(t)}{W_0}\,dt. \tag{28}$$

It is not difficult to see that for $c_2>c_2^*$, we have $y(x)\to\infty$ and $y'(x)\to c_2-c_2^*>0$ when $x\to\infty$. Similarly, for $c_2<c_2^*$ we have $y(x)\to-\infty$ and $y'(x)\to c_2-c_2^*<0$ when $x\to\infty$. This concludes the proof.

Remark 1. Under the hypothesis (10), for each solution $y=y(x)$ of (1), there exists

$$\lim_{x\to\infty}\frac{y(x)}{x}=\text{finite}. \tag{29}$$

If $y=y(x)$ is unbounded, this limit is $\neq 0$ (use the part ii)of Lemma 4 and Hospital rule), conversely being obviously true. If $y=y(x)$ is a bounded solution, this limit is zero, but conversely is also true.

Remark 2. Denoting $y(x)/x=z(x)$ for $x>0$, we can say that under the hypothesis (10), every solution of the equation

$$z''+\frac{2}{x}z'+Q(x)z=\frac{1}{x}f(x),\qquad x>0 \tag{30}$$

has finite limit for $x\to\infty$; see also ([2], Theorem 3).

In order to prove that the graph of each solution of (1) has asymptote for $x\to\infty$, we consider the conditions

$$\int_0^\infty x^2\,|\,Q(x)\,|\,dx<\infty,\qquad \int_0^\infty x\,|\,f(x)\,|\,dx<\infty. \tag{31}$$

T h e o r e m. *Under the hypothesis* (31), *the graph of each solution of the Eq.* (1) *has asymptote for* $x\to\infty$: *horizontal — for the bounded, oblique — for the unbounded solutions. If* $y=\alpha x+\beta$ *is the equation of the asymptote to the graph of the solution* $y=y(x)$, *the correspondence* $y=y(x)\leftrightarrow y=\alpha x+\beta$ *is one-to-one ;* α *and* β *may take any real values.*

P r o o f. Remembering the part i) of the Lemma 4, it suffices to consider only the case when $y=y(x)$ is an unbounded solution of the Eq. (1) —(31). By the

relation (29), the ratio $y(x)/x$ has finite limit $\alpha \neq 0$ when $x \to \infty$. Denoting $y(x) - \alpha x = u(x)$, we have to prove that this function has finite limit for $x \to \infty$. We remark that $u = u(x)$ satisfies the relations

$$u'' + Q(x)u = f(x) - \alpha x Q(x), \quad x \geqslant 0; \quad \lim_{x \to \infty} \frac{u(x)}{x} = 0. \tag{32}$$

Applying to the above obtained equation the Remark 1 and the part i) of the Lemma 4, we get that $u = u(x)$ is bounded for $x \geqslant 0$ and consequently has finite limit for $x \to \infty$. Denoting

$$\lim_{x \to \infty} u(x) = \beta = \text{finite} \tag{33}$$

we have that the straight line of the equation $y = \alpha x + \beta$ is asymptote to the graph of the given solution $y = y(x)$. If suppose that the graph of the solution $y = \tilde{y}(x)$ of (1) has the same asymptote $y = \alpha x + \beta$, then it follows $\tilde{y}(x) \equiv y(x)$. Indeed, from the relation

$$\lim_{x \to \infty} \frac{\tilde{y}(x) - y(x)}{x} = 0 \tag{34}$$

we deduce that $\tilde{y}(x) - y(x)$ is a bounded solution of the equation $y'' + Q(x)y = 0$. On the other hand, from $\tilde{y}(x) - \alpha x \to \beta$, $y(x) - \alpha x \to \beta$ as $x \to \infty$, we have $\tilde{y}(x) - y(x) \to 0$ as $x \to \infty$ and consequently $\tilde{y}(x) \equiv y(x)$. In other words, the correspondence solution $\leftrightarrow$ asymptote is one-to-one. Now, let a and b be arbitrarily choosen real numbers. Consider again the solutions $y_1(x)$ and $y_2(x)$ used in the Lemma 4 of the homogeneous equation $y'' + Q(x)y = 0$ and denote by $y_0(x)$ the solution of (1) for which $y_0(x) \to 0$ when $x \to \infty$ (Lemma 3). Remarking that

$$b_2 = \lim_{x \to \infty} (y_2(x) - x) \tag{35}$$

exists and is finite (see the relation (33)), it follows that the straight line $y = ax + b$ is asymptote to the graph of the solution $y(x) = y_0(x) + (b - ab_2)y_1(x) + ay_2(x)$ of the Eq. (1). This concludes the proof.

Remark 3. A similar proposition may be formulated relatively to the general equation of the second order

$$y'' + P(x)y' + Q(x)y = f(x), \quad x \geqslant 0 \tag{36}$$

which may be reduced by a simple substitution to the form (1).

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ECUAŢIA $y'' + Q(x)y = f(x)$ CU Q ŞI f FUNCŢII „MICI"

(Rezumat)

În cele patru leme se stabilesc proprietăţi generale ale soluţiilor ecuaţiei (1), în ipoteza foarte naturală (10). În ipoteza ceva mai restrictivă (31), se dovedeşte că graficul oricărei soluţii are asimptotă—orizontală sau oblică—pentru $x \to \infty$.

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THE STUDY OF A CLASS OF MULTIVOC DIFFERENTIAL EQUATIONS

BY

CĂTĂLINA DAVIDEANU

The purpose of our paper is to study the dependence of a multivoc differential equation (E) and of the corresponding approximative differential equation (E^ε) of the right-side. It is a sequence of the article [3].

We consider the multivoc differential equation

$$\text{(E)} \qquad y'(t)+(F+\beta)(y(t))\ni f(t) \quad \text{a.e.} \quad f\in\,]0,\,T[,$$

with the initial condition

$$\text{(1)} \qquad y(0)=y_0\in D(\beta).$$

The elements which occur here have the following properties :

(i) $F:\mathbf{R}^n\to\mathbf{R}^n$ is a local Lipschitz function which satisfies the condition : there exists $D_1, D_2\in\mathbf{R}_+^*$ so that :

$$\text{(2)} \qquad (Fy,\,y)\geqslant -D_1\|y\|^2-D_2 \quad \text{for every} \quad y\in\mathbf{R}^n\,;$$

(ii) $\beta : D(\beta)\subset\mathbf{R}^n\to\mathbf{R}^n$ is a maximal monotone operator with $0\in\beta(0)$;

$$\text{(iii)} \qquad f\in W^{1,1}(0,\,T\,;\,\mathbf{R}^n).$$

In this assumptions we may demonstrate

L e m m a 1. *If* $f\in W^{1,2}(0,\,T\,;\,\mathbf{R}^n)$ *and* $\{f^k\}_{k\in\mathbf{N}}\subset W^{1,2}(0,\,T\,;\,\mathbf{R}^n)$ *is a sequence such that*

$$\text{(3)} \qquad f^k\to f \quad \textit{weakly in} \quad W^{1,2}(0,\,T\,;\,\mathbf{R}^n), \quad \textit{for} \quad k\to\infty,$$

then $\{y^k(t)\}_{k\in\mathbf{N}}=\{y(t,\,y_0,\,f^k)\}_{k\in\mathbf{N}}$, *the sequence of the solutions of* (E) *corresponding to* $\{f^k\}_{k\in\mathbf{N}}$, *admits a sequence* $\{y^{k_j}\}_{j\in\mathbf{N}}$ *such that*

$$\text{(4)} \qquad \begin{aligned} y^{k_j}=y(.,\,y_0,\,f^{k_j})\to y=y(.\,,\,y_0,\,f) \ \textit{weakly in}\ C([0,\,T[\,;\,\mathbf{R}^n) \\ \textit{strongly in}\ W^{1,2}(0,\,T\,;\,\mathbf{R}^n), \end{aligned}$$

for $k_j\to\infty$.

P r o o f. Any function y^k satisfies the equation

$$\text{(E}^k\text{)} \qquad \frac{\mathrm{d}y^k(t)}{\mathrm{d}t}+(F+\beta)(y^k(t))\ni f^k(t) \quad \text{a.e.} \quad f\in\,]0,\,T[,$$

with the initial conditon

$$y^k(0)=y_0\in D(\beta).$$

We multiply scalar the equation (E^k) by y^k. Taking account the properties of F and β we get

$$\frac{d}{dt}\|y^k(t)\|^2\leqslant(2D_1+1)\|y^k(t)\|^2+2D_2+\|f^k(t)\|^2. \tag{5}$$

By integrating the relation (5) on the closed interval $[0, t]$ it becomes

$$\|y^k(t)\|^2\leqslant\|y_0\|^2+(2D_1+1)\int_0^t\|y^k(s)\|^2ds+2D_2T+\|f^k\|^2_{W^{1,2}(0,T;\mathbf{R}^n)}. \tag{5'}$$

Gronwall's lemma leads to

$$\|y^k\|_{C([0,T];\mathbf{R}^n)}\leqslant C,\quad \forall k\in\mathbf{N}, \tag{6}$$

where

$$C=\sqrt{D_3}\exp\left(\tfrac{1}{2}D_4T\right),\quad D_3=\|y_0\|^2+2D_2T+\|f^k\|^2_{W^{1,2}(0,T,\mathbf{R}^n)},\quad D_4=2D_1+1.$$

Substracting the relations satisfied by $y^k(t+h)$, $t\in]0, T-h[$ and $y^k(t)$ we get

$$\frac{d}{dt}\|y^k(t+h)-y^k(t)\|^2\leqslant(2L_c+1)\|y^k(t+h)-y^k(t)\|^2+\|f^k(t+h)-f^k(t)\|^2; \tag{7}$$

we have used the monotony of β and the fact that F is a local Lipschitz function.

From the existence Theorem 1 (see [3]) we have the inequality

$$\left\|\frac{d^+y^k(0)}{dt}\right\|\leqslant\|(f^k(0)-(F+\beta)(y_0))^0\|=C_1. \tag{8}$$

Integrating the relation (7) on the interval $[0, T-h]$ and using (8) we obtain

$$\begin{aligned}\|y^k(t+h)-y^k(t)\|^2\leqslant C_1+(2L_c+1)\int_0^{T-h}\|y^k(t+h)-y^k(t)\|^2dt+\\ +\int_0^{T-h}\|f^k(t+h)-f^k(t)\|^2dt.\end{aligned} \tag{7'}$$

We divide the above relation by h^2, make h tend to zero, apply Gronwall's lemma and get

$$\left\|\frac{dy^k}{dt}\right\|_{C([0,T];\mathbf{R}^n)}\leqslant C,\quad \forall k\in\mathbf{N}. \tag{9}$$

The Ascoli-Arzelà's theorem assures the existence of an element $y_1^* \in C([0, T]; \mathbf{R}^n)$ and a subsequence $\{y^{k_i}\}_{i \in \mathbf{N}}$ such that

$$y^{k_i} \to y_1^* \quad \text{strongly in} \quad C([0, T]; \mathbf{R}^n), \quad \text{for} \quad k_i \to \infty. \tag{10}$$

There results, from the relation (10), the existence of a subsequence $\{y^{k_j}\}_{j \in \mathbf{N}}$ and an element $\mu \in L^2(0, T; \mathbf{R}^n)$ such that

$$\frac{dy^{k_j}}{dt} \to \mu_1 = \frac{dy_1^*}{dt} \quad \text{weakly in } L^2(0, T; \mathbf{R}^n), \quad \text{for} \quad k_j \to \infty. \tag{11}$$

The relation (10) and (11) imply

$$y^{k_j} \to y_1^* \quad \text{weakly in } W^{1,2}(0, T; \mathbf{R}^n), \quad \text{for} \quad k_j \to \infty. \tag{12}$$

The continuity of F implies

$$Fy^{k_j} \to Fy_1^* \text{ in } \mathbf{R}^n, \quad \text{for} \quad k_j \to \infty. \tag{13}$$

The relations (3), (13) and (12) lead to the convergence

$$f^{k_j} - \frac{dy^{k_j}}{dt} - Fy^{k_j} \to f - \frac{dy_1^*}{dt} - Fy_1^*, \quad \text{weakly in} \quad L^2(0, T; \mathbf{R}^n), \text{ for } k_j^* \to \infty. \tag{14}$$

We note that the multivoc operator β is demicontinuous on $D(\beta)$ and it keeps the same property on $L^2(0, T; \mathbf{R}^n)$; with this observation from the equation (E^k) we get

$$f - \frac{dy_1^*}{dt} - Fy_1^* \in \beta(y_1^*),$$

that is $y_1^* = y(t, y_0, f)$, the unique solution of the multivoc differential equation (E). This equality results from the oneness of the solution (see [3]).

Further on, for every $\varepsilon > 0$, we construct the approximative differential equation

$$y'(t) + (F^\varepsilon + \beta^\varepsilon)(y(t)) = f(t) \quad \text{a.e.} \quad f \in]0, T[, \tag{E^ε}$$

with the initial condition

$$y(0) = y_0 \in \mathbf{R}^n,$$

where

$$\beta^\varepsilon(y) = \int_{\mathbf{R}^n} (\beta_\varepsilon(y - \varepsilon^2\theta) - \beta_\varepsilon(-\varepsilon^2\theta))\rho(\theta)d\theta + \beta_\varepsilon(0), \tag{15}$$

$$\beta_\varepsilon = \varepsilon^{-1}(I - (I + \varepsilon\beta)^{-1}), \tag{16}$$

$$F^\varepsilon(y) = \int_{\mathbf{R}^n} F(y - \varepsilon^2\theta)\rho(\theta)d\theta, \tag{17}$$

with $\rho: \mathbf{R}^n \to \mathbf{R}^n$ a function C_0^∞-mollifier on $\mathbf{R}^n$.

The function F^ε is also local Lipschitz and β^ε is a monotone operator too.

For $\varepsilon>0$ and $y_0\in\mathbf{R}^n$, $f\in W^{1,2}(0, T; \mathbf{R}^n)$ the conditions of the Theorem 1 from [3] are satisfied and therefore the equation (E^ε) admits a unique solution noted $y^\varepsilon(t) = y(t, y_0, f, F^\varepsilon, \beta^\varepsilon)$.

We may demonstrate the following

L e m m a 2. *The sequence of solutions $\{y^\varepsilon\}_{\varepsilon>0}$ of the approximative differential equations (E^ε) admits a subsequence $\{y^{\varepsilon_j}\}_{j\in\mathbf{N}}$ for which there are the convergences*

$$y^{\varepsilon_j}=y(\cdot, y_0, f, F^\varepsilon, \beta^\varepsilon)\to y=y(\cdot, y_0, f, F, \beta)$$

strongly in $C([0, T]; \mathbf{R}^n)$, weakly in $W^{1,2}(0, T; \mathbf{R}^n)$, for $\varepsilon_j\to 0$.

P r o o f. We may suppose, that F^ε satisfies a condition of type (2) with the constants $A_1, A_2\in\mathbf{R}_+^*$.

Multiplying scalar the equation (E^ε) satisfied by y^ε with y^ε and using the properties of F^ε and β^ε we get

$$\frac{d}{dt}\|y^\varepsilon(t)\|^2\leqslant(2A_1+1)\|y^\varepsilon(t)\|^2+2A_2+\|f(t)\|^2. \tag{18}$$

We integrate relation (18) on the closed interval $[0, t]$, apply the Gronwall's lemma and obtain

$$\|y^\varepsilon\|_{C([0, T]; \mathbf{R}^n)}\leqslant C, \qquad \forall\varepsilon>0, \tag{18'}$$

where

$$C=\sqrt{A_3}\exp\left(\tfrac{1}{2}A_4T\right), \quad A_3=\|y_0\|^2+2A_2T+\|f\|^2_{W^{1,2}(0, T; \mathbf{R}^n)}, \quad A_4=2A_1+1.$$

An analougous reasoning to the one in Lemma 1 leads to

$$\left\|\frac{dy^\varepsilon}{dt}\right\|_{C([0, T]; \mathbf{R}^n)}\leqslant C_2. \tag{19}$$

The relations (18′) and (19) allow us to apply the Ascoli-Arzelà's theorem which assures the existence of an element $y_2^*\in C[0, T]; \mathbf{R}^n)$ and a subsequence $\{y^{\varepsilon_i}\}$ so that

$$y^{\varepsilon_i}\to y_2^* \quad \text{strongly in} \quad C([0, T]; \mathbf{R}^n), \quad \text{for} \quad \varepsilon_i\to 0. \tag{20}$$

We may show (see Lemma 1) that there exists an element $\mu_2\in L^2(0, T; \mathbf{R}^n)$ such that

$$\frac{dy^{\varepsilon_j}}{dt}\to\mu_2=\frac{dy_2^*}{dt} \quad \text{weakly in} \quad L^2(0, T; \mathbf{R}^n) \quad \text{for} \quad \varepsilon_j\to 0. \tag{21}$$

The relations (20) and (21) imply the convergence

$$y^{\varepsilon_j}\to y_2^* \quad \text{weakly in} \quad W^{1,2}(0, T; \mathbf{R}^n), \quad \text{for} \quad \varepsilon_j\to 0. \tag{22}$$

Using the properties of F^ε there results

$$F^{\varepsilon_j}y^{\varepsilon_j}\to Fy_2^* \quad \text{in} \quad \mathbf{R}^n, \quad \text{for} \quad \varepsilon_j\to 0. \tag{23}$$

The relations (20)—(23) lead to

$$\beta^{\varepsilon_j} y^{\varepsilon_j} = f - \frac{dy^{\varepsilon_j}}{dt} - F^{\varepsilon_j} y^{\varepsilon_j} \to f - \frac{dy_2^*}{dt} - F y_2^* \tag{24}$$

weakly in $L^2(0, T; \mathbf{R}^n)$, for $\varepsilon_j \to 0$.

The convergence (20) and the properties of the operator $J_\varepsilon = (I + \varepsilon\beta)^{-1}$ imply

$$J_{\varepsilon_j} y^{\varepsilon_j} \to y_2^* \quad \text{strongly in} \quad L^2(0, T : \mathbf{R}^n). \tag{25}$$

The convergences (25) and (26) result in

$$\beta_{\varepsilon_j} y^{\varepsilon_j} \in \beta(J_{\varepsilon_j} y^{\varepsilon_j}) \tag{26}$$

and the fact that β is an operator demicontinuous leads to

$$f - \frac{dy_2^*}{dt} - F y_2^* \in \beta(y_2^*), \tag{27}$$

which means that $y_2^* = y(\cdot, y_0, f, F, \beta)$, is the unique solution of the multivoc differential equation (E).

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STUDIUL UNEI CLASE DE ECUAŢII DIFERENŢIALE MULTIVOCE

(Rezumat)

Se consideră ecuaţia diferenţială multivocă (E) cu condiţia iniţială (1). Se studiază dependenţa de membrul drept a ecuaţiei diferenţiale multivoce (E) cît şi a ecuaţiei aproximative corespunzătoare (E^ε). Rezultatele obţinute sînt prezentate în Lemele 1 şi 2.

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CONTINUOUS SELECTIONS FOR MULTIFUNCTIONS SATISFYING THE CARATHÉODORY TYPE CONDITIONS. THE CAUCHY PROBLEM ASSOCIATED TO A MULTI-VALUED EQUATION

BY

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1. Introduction. By analogy to [1], in this Note one proves an existence theorem for a continuous selection for each of the functions $(x, y) \to F(x, y, z(x, y))$, where F is a multifunction defined on a compact set in $\mathbb{R}^{n+2}$ and valued in the set of nonconvex compact sets of $\mathbb{R}^n$, relatively to a given family of continuous functions $(x, y) \to z(x, y)$, F satisfying the Carathéodory type conditions. The main result (Theorem 2), is an extension of Theorem 1 [2], where F is a continuous multifunction. The results in [1] remain true in this case. As consequence, one obtains an existence theorem of an absolutely continuous solution for the Cauchy problem associated to the multi-valued equation $\partial^2 z/\partial x \partial y \in F(x, y, z)$ [3], [4].

2. Preliminaries. We recall the main notations and results given in [2]. Let be the multifunction $F: D \times B \to \operatorname{comp} X$, $D = [0, a] \times [0, b]$, $B \subset \mathbb{R}^n$ is the closed ball centered in origin and of radius $c = M_1 + Mab$, M_1 given by (5), M given by (6), and $X \subset \mathbb{R}^n$ is the closed ball centered in origin and of radius M. Obviously, X is a compact space in the metric d induced on X by the norm of $\mathbb{R}^n$. The family of nonempty compact sets in X, denoted comp X, is a compact metric space for the Hausdorff metric H on comp X induced by the metric d.

Let $\mathcal{C}(D; \mathbb{R}^n)$ the Banach space of continuous functions from D to $\mathbb{R}^n$ and $\mathcal{L}^1(D; \mathbb{R}^n)$ be the Banach space of equivalence classes of Lebesgue integrable functions defined on D and valued in $\mathbb{R}^n$.

Consider the following hypotheses :

(H_0) The curve $\gamma: y = \varphi(x)$, $0 \leqslant x \leqslant a$, is defined by the function φ, $\varphi \in C^1([0, a]; \mathbb{R})$, $\varphi' \in AC([0, a]; \mathbb{R})$, satisfying the relations

$$\varphi(0) = 0, \quad \varphi(a) = b, \quad \varphi'(x) > 0, \quad 0 \leqslant x \leqslant a. \tag{1}$$

We remark that the equation of the curve γ can be written under the form $\gamma: x = \psi(y)$, $0 \leqslant y \leqslant b$, $\psi = \varphi^{-1}$, submitted to the conditions

$$\psi(0) = 0, \quad \psi(b) = a, \quad \psi'(y) > 0, \quad 0 \leqslant y \leqslant b. \tag{2}$$

(H_1) The functions $R, P, R' \in AC([0, a]; \mathbb{R}^n)$, where $AC([\alpha_1, \alpha_2]; \mathbb{R}^n)$ is the space of absolutely continuous functions $f: [\alpha_1, \alpha_2] \to \mathbb{R}^n$, endowed with the norm

$$|||f||| = \sup_{t \in [\alpha_1, \alpha_2]} \|f(t)\| + \int_{\alpha_1}^{\alpha_2} \|f'(t)\| dt.$$

The function $Q: [0, a] \to \mathbb{R}^n$ is defined by

$$Q(x) = \frac{1}{\varphi'(x)} [R'(x) - P(x)], \quad 0 \leqslant x \leqslant a. \tag{3}$$

(H_2) The function $\alpha: D \to \mathbb{R}^n$, defined by

$$\alpha(x, y) = R(x) - \int_{y}^{\varphi(x)} Q(\psi(v)) dv, \tag{4}$$

is bounded, and therefore is $M_1 > 0$ such that

$$\|\alpha(x, y)\| \leqslant M_1, \quad (x, y) \in D. \tag{5}$$

It follows that α is absolutely continuous on D; $\alpha \in C^*(D; \mathbb{R}^n)$, [5, §565 – §568].

Let K be the set of absolutely continuous functions $z: D \to \mathbb{R}^n$, $z \in C^*(D; \mathbb{R}^n)$, satisfying (5), (6), (7) where

$$\left\| \frac{\partial^2 z(x, y)}{\partial x \partial y} \right\| \leqslant M, \quad \text{a.e.} \quad (x, y) \in D, \quad M > 0, \tag{6}$$

and

$$z(x, \varphi(x)) = R(x), \quad \frac{\partial z}{\partial x}(x, \varphi(x)) = P(x), \quad 0 \leqslant x \leqslant a. \tag{7}$$

P r o p o s i t i o n 1. *The set K is a nonempty, convex and compact subset of $\mathcal{C}(D; \mathbb{R}^n)$.*

The relation $z \in K$ implies $z \in C(D; \mathbb{R}^n)$. We observe that $\frac{\partial^2 z}{\partial x \partial y}(x, y)$ exists a.e. $(x, y) \in D$, as $z \in C^*(D; \mathbb{R}^n)$ [5]. Let $N(x, y)$, $N'(x = \psi(y), y)$, $N''(x, \psi(x))$, with $y \leqslant \varphi(x)$, three points in D and the triangle $T(x, y) = \{(u, v) \mid \psi(v) \leqslant u \leqslant x, y \leqslant u \leqslant \psi(x)\} \subset D$, $(x, y) \in D$.

Insegrating $\frac{\partial^2 z(x, y)}{\partial x \partial y}$ over $T(x, y)$ and using (3) and (7) it follows

$$\begin{aligned} z(x, y) &= R(x) - \int_{y}^{\varphi(x)} Q(\psi(v)) dv - \int_{y}^{\varphi(x)} dv \int_{\psi(v)}^{x} \frac{\partial^2 z(u, v)}{\partial u dv} du = \\ &= \alpha(x, y) - \iint_{T(x, y)} \frac{\partial^2 z(u, v)}{\partial u \partial v} du dv. \end{aligned} \tag{8}$$

Remark. The relation $z \in K$ implies $(x, y, z(x, y)) \in D \times B$ for each $(x, y) \in \in D$. Therefore, each $z \in K$ generates a map $(x, y) \to F(x, y, z(x, y))$ of D into comp X, denoted $G(z)$.

This function G is associated to the multifunction F by the relation

$$G(z)(x, y) = F(x, y, z(x, y)), \quad (x, y) \in D. \tag{9}$$

The main theorem in [2] on the existence of a continuous selection it the following

Theorem 1. *If $F : D \times B \to \text{comp } X$ is a continuous multifunction, then there exists a continuous function $g : K \to \mathcal{L}^1(D ; \mathbb{R}^n)$ such that, for every $z \in K$, $g(z)(x, y) \in G(z)(x, y)$, a.e. $(x, y) \in D$.*

3. Continuous Selections, The Measurable Case. The Theorem 1 can be extended to multifunctions defined on $D \times B$ and valued in comp X, and satisfying the Carathéodory type conditions, similar to [1].

Theorem 2. *Suppose $F : D \times B \to \text{comp } X$ satisfies the following hypotheses :*

i) *for each $z \in B$, $(x, y) \to F(x, y, z)$ is measurable in D;*

ii) *for each $(x, y) \in D$, $z \to F(x, y, z)$ is continuous in B.*

Then, there exists a continuous function $g : K \to \mathcal{L}^1(D : \mathbb{R}^n)$ such that, for each $z \in K$, $g(z)(x, y) \in G(z)(x, y)$, a.e. $(x, y) \in D$.

Proof. One uses the proof of Theorem 2, wich rests upon the fololwing two properties of the multifunction $F : D \times B \to \text{comp } X$, which are direct consequences of the assumed continuity :

1° the family $\{F(x, y, \cdot)\}$, $(x, y) \in D$ of maps of B into comp X is uniformly equicontinuous ;

2° for each $z \in \mathcal{C}(D : \mathbb{R}^n)$ the map $G(z)(x, y) \to F(x, y, z(x, y))$ is measurable in D.

We shall show that the hypotheses of Theorem 2 induce in F essentially the same properties. It is a consequence of the following two lemmas.

The Lemma 1 is a similar result of the theorems of Scorza-Dragoni type for multifunctions [1], [6], [7].

Lemma 1. *Suppose $F : D \times B \to \text{comp } X$ satisfies the hypotheses of Theorem 2. Then, for each $\varepsilon > 0$, there exists a closed set $E \subset D$ with $\mu(D - E) \leqslant \varepsilon$, such that the family $\{F(x, y, \cdot)\}$, $(x, y) \in E$, of maps of B into comp X is uniformly equicontinuous.*

Proof. Let be the functions $\delta_n : D \to R$, $n \geqslant 1$, defined by

$$\delta_n(x, y) = \sup \left\{ \rho > 0 : \|u - v\| < \rho \Rightarrow H(F(x, y, u), F(x, y, v)) < \frac{1}{n}, u, v \in B \right\}. \tag{10}$$

To verify the conclusion of this Lemma, one shows that for each $\varepsilon > 0$, there exists a closed set $E \subset D$, wich $\mu(D - E) \leqslant \varepsilon$, such that the restriction δ_n / E it continuous.

Let $A = \{z_m\}_{m \geqslant 1}$ be a dense subset of B. The hypothesis i) implies the existence of a closed set $L \subset D$ with $\mu(D - L) \leqslant \varepsilon/2$ such that the restriction to L of every function $(x, y) \to F(x, y, z_m)$ is continuous. Then, each restriction $\delta_{n/L}$ is upber semicontinuous, hence measurable.

Indeed, suppose, by reductio ad absurdum, that for (fixed) $n \geqslant 1$ there exists a point $(x_0, y_0) \in L$, a constant $\nu > 0$, and a sequence $\{(x_k, y_k)\}_{k \geqslant 1}$ convergent to (x_0, y_0) in D, such that

$$\delta_n(x_k, y_k) \geqslant \delta_n(x_0, y_0) + \nu, \quad \text{for all } k \geqslant 1. \tag{11}$$

Fet v_1, v_2 be points in B with

$$\delta_n(x_0, y_0) \leqslant \|v_1 - v_2\| \leqslant \delta_n(x_0, y_0) + \tfrac{3}{4}\nu, \tag{12}$$

such that, for some $\Delta > 0$,

$$H(F(x_0, y_0, v_1), F(x_0, y_0, v_2)) = \frac{1}{n} + \Delta. \tag{13}$$

Then, there exists points z_{m_1}, z_{m_2} in A and a point $(x_k, y_k) \in L$ such that

$$\|v_1 - z_{m_1}\| < \frac{\nu}{8}, \quad \|v_2 - z_{m_2}\| < \frac{\nu}{8}. \tag{14}$$

Also, by hypothesis i) we have

$$\begin{cases} H(F(x_0, y_0, v_1), F(x_0, y_0, z_{m_1})) < \dfrac{\Delta}{8}, \\ H(F(x_0, y_0, v_2), F(x_0, y_0, z_{m_2})) < \dfrac{\Delta}{8}, \end{cases} \tag{15}$$

and

$$\begin{cases} H(F(x_0, y_0, z_{m_1}), F(x_k, y_k, z_{m_1})) < \dfrac{\Delta}{8}, \\ H(F(x_0, y_0, z_{m_2}), F(x_k, y_k, z_{m_2})) < \dfrac{\Delta}{8}. \end{cases} \tag{16}$$

According to (11), (12), (14) one obtains

$$\begin{aligned} [z_{m_1} - z_{m_2}\| &\leqslant [z_{m_1} - v_1\| + \|v_1 - v_2\| + \|v_2 - z_{m_2}\| < \frac{\nu}{4} + \|v_1 - v_2\| \leqslant \\ &\leqslant \delta_n(x_0, y_0) + \nu \leqslant \delta_n(x_k, y_k). \end{aligned} \tag{17}$$

The relat ons (10) and (17) imply

$$H(F(x_k, y_k, z_{m_1}), F(x_k, y_k, z_{m_2})) < \frac{1}{n}. \tag{18}$$

Then, from (15), (16), (18) one obtains

$$\begin{aligned} H(F(x_0, y_0, v_1), F(x_0, y_0, v_2)) &\leqslant H(F(x_0, y_0, v_1), F(x_0, y_0, z_{m_1})) + \\ &+ H(F(x_0, y_0, z_{m_1}), F(x_k, y_k, z_{m_1})) + H(F(x_k, y_k, z_{m_1}), F(x_k, y_k, z_{m_2})) + \\ &+ H(F(x_k, y_k, z_{m_2}), F(x_0, y_0, z_{m_2})) + H(F(x_0, y_0, z_{m_2}), F(x_0, y_0, v_2)) < \\ &< H(F(x_k, y_k, z_{m_1}), F(x_k, y_k, z_{m_2})) + \frac{\Delta}{2} < \frac{1}{n} + \frac{\Delta}{2}, \end{aligned} \tag{19}$$

wich is in contradiction to (13).

It follows that there exists a closed set $E \subset L$ with $\mu(E-L) \leqslant \varepsilon/2$ such that each restriction $\delta_{n/E}$ is continuous. Obviously, by construction, $\mu(D-E) \leqslant \varepsilon$.

L e m m a 2. *Suppose* $F: D \times B \to \text{comp } X$ *satisfies the hypotheses of Theorem* 2. *Then, for each* $z \in \mathcal{C}(D: \mathbb{R}^n)$, $G(z): (x, y) \to F(x, y, z(x, y))$ *is measurable in* D.

P r o o f. Let $z \in \mathcal{C}(D; \mathbb{R}^n)$ be given, and let $\{z_n\}_{n \geqslant 1}$ be a sequence of piecewise constant maps in D that converges so z uniformly in D. We need only to show that, for every $\varepsilon > 0$, there exists a closed set $E_0 \subset D$ with $\mu(D-E_0) \leqslant \varepsilon$ such that $G(z)/E_0$ is continuous.

Taking inso account Lemma 1,there is a closed set $E \subset D$ with $\mu(D-E) \leqslant \leqslant \varepsilon/2$ such that the sequence $\{G(z_n)\}_{n \geqslant 1}$ converge to $G(z)$ uniformly in E. Since each $G(z_n)$ is measurable in D and hence in E, there exists a closed set $E_0 \subset E$ with $\mu(E-E_0) \leqslant \varepsilon/2$ such that each restriction $G(z_n)/E_0$ is continuous. This implies that $G(z)/E_0$ is continuous.

To prove the Theorem 2, one shows an increasing sequence $\{E_n\}_{n \geqslant 1}$ of closed sets $E_n \subset D$ with $\mu(D-E_n) \leqslant 2^{-n}$ such that each restriction $F/E_n \times B$ is continuous and one defines for each $n \geqslant 1$,

$$F^n(x, y, z) = \begin{cases} F(x, y, z), & (x, y, z) \in E_n \times B. \\ 0 \quad , & (x, y, z) \in D - E_n \times B. \end{cases}$$

Then, as a consequence of Theorem 1, that there exists, for every $n \geqslant 1$ a continuous map $g^n: K \to \mathcal{L}^1(D; \mathbb{R}^n)$ such that for each $z \in K$, $g^n(z)(x, y) \in \in F(x, y, z(x, y))$ a.e. $(x, y) \in E_n$.

Let $A_1 = E_1$ and $A_{n+1} = E_{n+1} - E_n$ for every $n \geqslant 1$ so that $E_n = \bigcup_{k=1}^{n} A_k$, and define $g(z)$, for each $z \in K$, by setting

$$g(z)_{/A_n} = g^n(z) \quad \text{and} \quad g(z)_{/D - \bigcup_{n=1}^{\infty} A_n} = 0.$$

Obviously, g maps K into $\mathcal{L}^1(D; \mathbb{R}^n)$ and, for each $z \in K$, $g(z)(x, y) \in (F(x, y, z(x, y))$ a.e.$(x, y) \in D$. Also, $g: K \to \mathcal{L}^1(D: \mathbb{R}^n)$ is continuous since, for any $z \in K$ and $w \in K$

$$\int_D \|g(z)(x, y) - g(w)(x, y)\| \mathrm{d}x\mathrm{d}y = \int_{D-E_n} \|g(z)(x, y) - g(w)(x, y)\| \mathrm{d}x\mathrm{d}y +$$

$$+ \int_{E_n} \|g(z)(x, y) - g(w)(x, y)\| \mathrm{d}x\mathrm{d}y \leqslant 2^{-n+1} M + \sum_{k=1}^{n} \int_{A_k} \|g^k(z)(x, y) - g^k(w)(x, y)\| \mathrm{d}x\mathrm{d}y,$$

whatever $n \geqslant 1$ and each g^k is continuous, by construction.

4. The Cauchy Problem Associated to a Multi-valued Equation. Consider the multi-valued equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \quad (x, y) \in D, \quad z \in B, \tag{20}$$

where $F: D \times B \to \text{comp } X$.

The Cauchy problem associated to the Eq. (20) is defined in [3], [4] and consists in the determination of an absolutely continuous function satisfying (20) and (7) in the hypotheses (H_0), (H_1), (H_2).

As in [2] we establish the following

Theorem 3. *Suppose* $F: D\times B\to \mathrm{comp}\, X$ *satisfies the hypotheses* i), ii) *of the Theorem* 2. *If the hypotheses* (H_0), (H_1), (H_2) *is fulfilled, the Cauchy problem* (20)+(7) *has at least an absolutely continuous solution* $\bar{z}: D\to\mathbb{R}^n$, $\bar{z}\in C^k(D;\mathbb{R}^n)$.

The proof uses the Theorem 2 and is similar to that given for Theorem 2 [2].

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SELECŢII CONTINUE PENTRU MULTIFUNCŢII CARE SATISFAC CONDIŢII DE TIP CARATHÉODORY.

PROBLEMA LUI CAUCHY ASOCIATĂ UNEI ECUAŢII MULTIVOCE

(Rezumat)

Se demonstrează o teoremă de existenţă a unei selecţii continue pentru fiecare din aplicaţiile $(x, y)\to F(x, y, z(x, y))$, unde F este o multifuncţie definită pe o mulţime compactă din $\mathbf{R}^{n+2}$ cu valori-mulţimi compacte, neconvexe din $\mathbf{R}^n$, relativ la o familie dată de funcţii continue $(x, y)\to z(x, y)$, $(x, y)\in D[0, a]\times[0, b]$, F satisfăcînd condiţii de tip Carathéodory. Folosind acest rezultat se demonstrează o teoremă de existenţă a unei soluţii absolut continue pentru problema lui Cauchy asociată ecuaţiei multivoce $\dfrac{\partial^2 z}{\partial x\partial y}\in F(x, y, z)$.

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OBSERVATIONS ON A HAMILTON-JACOBI EQUATION

BY

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1. Introduction. In this paper we will study the Hamilton-Jacobi equation

$$\varphi_t(x, t)+F(-D\varphi_x(t, x))+(Ax, \varphi_x(t, x))+\mu(\varphi(t, x))=g(x), \tag{1}$$

$$\varphi(0, x)=\varphi_0(x), \quad \text{in} \quad \mathbb{R}^+\times H,$$

where H is a real Hillbert space with the norm $|\cdot|$ and scalar product $(\cdot, \cdot)$, the unknown function φ is a real valued function defined on $[0, \infty)\times H$ and φ_0 and g are given and defined functions on H.

The hypotheses which we will work are :

(i) $F : U\to\mathbb{R}$ is a convex continuous function on a real Hilbert space U with the norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)_U$:

(ii) $D : H\to U$ is a linear continuous operator from H to U :

(iii) $A\subset H\times H$ is a maximal monotone (multivalued) operator ;

(iv) $\mu : \mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous and monotone increasing function.

For X a metric space, we denote by BUC (X) the space of real valued, bounded and uniformly continuous functions on X, endowed with the norm

$$\|f\|_b=\sup\ \{|f(x)|, x\in X\},$$

and by LIP (X) the space of Lipschitz funtions $f : X\to\mathbb{R}$.

We denote by $\overline{D(A)}$ the closure of the domain $D(A)$ of A and by Y the space BUC $(\overline{D(A)})$.

Eq. (1) with $\mu\equiv 0$:

$$\varphi_t(t, x)+F(-D\varphi_x(t, x))+(Ax, \varphi_x(t, x))=g(x), \tag{2}$$

$$\varphi(0, x)=\varphi_0(x) \quad \text{în} \quad \mathbb{R}^+\times H$$

was studied by V. B a r b u în [1]. This equation is related to optimal control problems governed by the equation

$$\begin{cases} y'(t)+Ay(t)\ni D^*u, & t\geqslant 0, \\ y(0)=x, \end{cases} \tag{3}$$

with the functional

$$\int_0^t (g(y(s)) + h(u(s))\mathrm{d}s + \varphi_0(y(t)),$$

where D^* is the adjoint of operator D and h: $U \to \overline{R}$ is the conjugate on the function F.

The function

$$\varphi(t, x) = (S(t)\varphi_0)(x) = \inf\left\{\int_0^t (g(y(s)) + h(u(s)))\mathrm{d}s + \varphi_0(x(t)); u \in L^1(0, t; U),\right. \tag{4}$$

$$\left. y(s) = y(s, x, u) \text{ is the solution of the Eq. (3)}\right\}, \qquad t \geqslant 0, \quad x \in \overline{D(A)}$$

may be viewed as a generalization solution (viscosity solution) to Eq. (2).

In [1] it was proved that, for g and φ_0 in a certain subset of Y, the function (4) is the solution to nonlinear evolution equation

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = \mathcal{A}^\circ\varphi & \text{in } \mathbb{R}^+ \\ \varphi(0) = \varphi_0, \end{cases}$$

where $\mathcal{A}^\circ\varphi = \mathcal{A}\varphi + g$ is a single-valued, m-dissipative operator in the space Y.

We recall that the operator $\mathcal{A}$ was defined as :

$$\mathcal{A} : D(\mathcal{A}) \to Y, \quad D(\mathcal{A}) = \{R(1)f;\ f \in Y\},$$

$$\mathcal{A}R(1)f = R(1)f - f, \quad \forall f \in Y,$$

where

$$(R(\lambda)f)(x) = \inf\left\{\int_0^\infty \mathrm{e}^{-\lambda t}(f(y(t)) + h(u(t)))\mathrm{d}t\ ;\ u \in L^1_{\mathrm{loc}}(\mathbb{R}_+; U)\ ;\ y(t) = y(t, x, u)\right.$$

$$\left. \text{is the solution of the Eq. (3)}\right\}.$$

Also, în [1] it was showed that $\overline{D(\mathcal{A})} = \overline{\mathcal{D}}$, where

$$\mathcal{D} = \left\{\varphi \in \mathrm{BUC}(\overline{D(A)}) \cap \mathrm{LIP}(\overline{D(A)})\ ;\ |\varphi(y(t, x, u) - \varphi(x)| \leqslant L_\varphi\left(t + \int_0^t \|u(s)\|\mathrm{d}s\right);\right. \tag{5}$$

$$\left. \forall t \in [0, 1];\ u \in L^1(0, 1;\ U);\ y(t) = y(t, x, u) \text{ is the solution of the Eq. (3)}\right\}.$$

Analogous to the Eq. (2), the generalized solution of the Eq. (1) may be viewed as the unique weak solution of the evolution equation

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = \mathcal{A}^\circ\varphi + \mathcal{B}\varphi & \text{in } \mathbb{R}^+, \\ \varphi(0) = \varphi_0, \end{cases} \tag{6}$$

where $\mathcal{B} : Y \to Y$ is defined by $(\mathcal{B}\varphi)(x) = -\mu(\varphi(x))$, $\forall x \in \overline{D(A)}$.

We will demonstrate that restricting the operator $\mathcal{A}$ to a subset of its domain, equivalently, considering for the Eq. (1) φ_0 and g in a subset of the space Y, the solution of the Eq. (6) can be expressed as a function of the semigroups generated by the operators $\mathcal{A}^\circ$ and $\mathcal{B}$ with the aid of the Lie-Trotter product formula.

We mention at the end of this introduction the result of Ph. Bénilan and S. Ismail from [3] which we have used in our proofs,

Proposition 1. *Let X be a Banach space, A an operator from X to itsef, m-dissipative with $\overline{D(A)}=X$.*

Then, the Lie-Trotter product formula is valid for any continuous and dissipative operator $B: X\to X$, that is

$$S^{A+B}(t)x=\lim_{n\to\infty}\left[S^A\left(\frac{t}{n}\right)S^B\left(\frac{t}{n}\right)\right]^n x, \qquad \forall x\in X,$$

uniformly in t on every compact interval of $[0,\infty)$, if and only if

$$\lim_{\varepsilon\searrow 0}\left(I-\lambda\,\frac{S^A(\varepsilon)-I}{\varepsilon}\right)^{-1}x=(I-\lambda A)^{-1}x, \qquad \forall x\in X, \qquad \lambda>0,$$

where $S^A(t):\overline{D(A)}\to\overline{D(A)}$, $t\geqslant 0$ is the semigroup generated by the operator A and defined by the Crandall-Liggett exponential formula.

2. Results. We denote by Z the closure of the set $\mathcal{D}$, defined in (5). The set $Z=\overline{\mathcal{D}}$ is a Banach space. Indeed, we observe at first that we can organize the set $\mathcal{D}$ as a linear subspace of the space Y. For $\varphi_1,\varphi_2\in\mathcal{D}$ and $\alpha_1,\alpha_2\in\mathbb{R}$, $\alpha_1\varphi_1+\alpha_2\varphi_2\in \mathrm{BUC}(\overline{D(A)})\cap \mathrm{LIP}(\overline{D(A)})$ and, moreover $L_{\alpha_1\varphi_1+\alpha_2\varphi_2}=|\alpha_1|L_{\varphi_1}+|\alpha_2|L_{\varphi_2}$. Hence, $\overline{\mathcal{D}}$ is a linear closed subspace in the Banach space Y and it results from this that $\overline{\mathcal{D}}$ is a complete subspace of Y.

We define the operator $\mathcal{A}_1: D(\mathcal{A}_1)\to Z$, as

$$D(\mathcal{A}_1)=\{R(1)f,\ f\in Z\},\quad \mathcal{A}_1R(1)f=R(1)f-f.$$

We have that $D(\mathcal{A}_1)\subset D(\mathcal{A})\subset Z$.

Lemma. *For $f\in\mathcal{D}$, $f_\varepsilon=(I-\varepsilon\mathcal{A})^{-1}f\to f$ for $\varepsilon\to 0$ in space Z.*

With the aid of this lemma, we will demonstrate that the operator $\mathcal{A}_1$ is densely defined in the space Z.

Proposition 2. $\overline{D(\mathcal{A}_1)}=Z$.

For $g\in Z$ we define the operator $\mathcal{A}_1^0: D(\mathcal{A}_1)\to Z$ by

$$\mathcal{A}_1^0\varphi=\mathcal{A}_1\varphi+g, \quad \forall\varphi\in D(\mathcal{A}_1).$$

Proposition 3. *The operator $\mathcal{A}_1^0$ is a single-valued, densely defined and m-dissipative operator in the space Z.*

This affirmation results from the definition of the operators $\mathcal{A}_1$ and $\mathcal{A}_1^\circ$, from the Proposition 2 and from the fact that $\mathcal{A}$ is a single-valued, m-dissipative operator in Y.

Now let $\mathcal{B}_1: Z\to Z$ be the operator defined by

$$(\mathcal{B}_1\varphi)(x)=-\mu(\varphi(x)), \quad \forall\varphi\in Z, \quad \forall x\in\overline{D(A)}.$$

Proposition 4. *The operator $\mathcal{B}_1$ is a continuous and dissipative operator on Z.*

The main result is the

T h e o r e m. *Assume φ_0 and g from Z. Then, for the evolution equation*

$$\frac{d\varphi}{dt}=\mathcal{A}_1^0+\mathcal{B}_1\varphi \quad in \quad \mathbb{R}^+,$$

$$\varphi(0)=\varphi_0,$$

associated with the Eq. (1), *we have the Lie-Trotter product formula*

$$S^{\mathcal{A}_1^0+\mathcal{B}_1}(t)\varphi_0=\lim_{n\to\infty}\left[S^{\mathcal{A}_1^0}\left(\frac{t}{n}\right)S^{\mathcal{B}_1}\left(\frac{t}{n}\right)\right]^n\varphi_0, \qquad \forall\varphi_0\in Z,$$

uniformly in t on every compact interval of $[0,\infty)$.

3. Proofs. P r o o f o f L e m m a. We will prove that $f_\varepsilon\to f$ in Z as $\varepsilon\to 0$. For $\varepsilon>0$ we have

$$f_\varepsilon(x)=R(\varepsilon^{-1})(\varepsilon^{-1}f)(x)=\inf\left\{\int_0^\infty e^{-t/\varepsilon}(\varepsilon^{-1}f(y_\varepsilon(t))+h(u(t)))dt,\ u_\varepsilon\in L^1_{loc}(\mathbb{R}^+;\ U),\right.$$

$$\left. y_\varepsilon \text{ is the solution of the Eq. (3)}\right\}.$$

For a constant function u_0 and the corresponding solution $y_0=y(t,x,u_0)$ of the Eq. (3), it results

$$f_\varepsilon(x)-f(x)\leqslant\int_0^\infty e^{-t/\varepsilon}(\varepsilon^{-1}f(y_0(t))+h(u_0))dt-f(x)=\int_0^\infty e^{-t/\varepsilon}\varepsilon^{-1}[f(y_0(t))-f(x)]dt+$$

(7)

$$+h(u_0)\int_0^\infty e^{-t/\varepsilon}dt\leqslant L_f\int_0^\infty \varepsilon^{-1}e^{-t/\varepsilon}\left(t+\int_0^t\|u_0\|ds\right)dt+\varepsilon h(u_0)=\varepsilon L_f(1+\|u_0\|)+$$

$$+\varepsilon h(u_0)\to 0 \qquad \text{for } \varepsilon\to 0, \qquad \text{uniformly in } x.$$

For $\varepsilon>0$ and all $x\in\overline{D(A)}$ there are y_ε, u_ε so that

$$f_\varepsilon(x)\geqslant\int_0^\infty e^{-t/\varepsilon}(\varepsilon^{-1}f(y_\varepsilon(t))+h(u_\varepsilon(t))df-\varepsilon.$$

Therefore

$$f(x)-f_\varepsilon(x)\leqslant f(x)-\int_0^\infty e^{-t/\varepsilon}(\varepsilon^{-1}f(y_\varepsilon(t))+h(u_\varepsilon(t)))dt+\varepsilon=$$

$$=\varepsilon+\int_0^\infty e^{-t/\varepsilon}\varepsilon^{-1}[f(x)-f(y_\varepsilon(t))]dt-\int_0^\infty e^{-t/\varepsilon}h(u_\varepsilon(t))dt\leqslant$$

(8)

$$\leqslant\varepsilon+L_f\int_0^\infty\varepsilon^{-1}e^{-t/\varepsilon}\left(t+\int_0^t\|u_\varepsilon(s)\|ds\right)dt+\int_0^\infty e^{-t/\varepsilon}|\,h(u_\varepsilon(t))\,|\,dt=$$

$$=\varepsilon(1+L_f)+L_f\int_0^\infty e^{-t/\varepsilon}\|\,u_\varepsilon(t)\|dt+\int_0^\infty e^{-t/\varepsilon}|h(u_\varepsilon(t))|dt\to 0$$

$$\text{for } \varepsilon\to 0, \text{ uniformly in } x,$$

because

$$\int_0^\infty e^{-t/\varepsilon} \mid h(u_\varepsilon(t)) \mid dt, \int_0^\infty e^{-t/\varepsilon} \|u_\varepsilon(t))\| dt \to 0 \quad \text{for} \quad \varepsilon \to 0,$$

according to the Proposition 2 from [1].

We infer by (7) and (8) that $\mid f(x) - f_\varepsilon(x) \mid \to 0$ for $\varepsilon \to 0$, uniformly in x; hence $f_\varepsilon \to f$ as $\varepsilon \to 0$ in space Z.

Proof of Proposition 2. Owing to $D(\mathcal{A}_1) \subset Z$ it is sufficiently to demonstrate that $\mathcal{D} \subset \overline{D(\mathcal{A}_1)}$. Let f be from $\mathcal{D}$. By Lemma we know that $f_\varepsilon = (I - \varepsilon\mathcal{A})^{-1} f \to 0$ as $\varepsilon \to 0$. Evidently $f_\varepsilon \in D(\mathcal{A})$, therefore $f_\varepsilon = R(1) g_\varepsilon$, $\varepsilon > 0$ with $g_\varepsilon \in Y$. Besides

$$g_\varepsilon = (I - \mathcal{A}) f_\varepsilon = (I - \mathcal{A})(I - \varepsilon\mathcal{A})^{-1} f = \frac{1}{\varepsilon}(I - \varepsilon\mathcal{A})(I - \varepsilon\mathcal{A})^{-1} f +$$

$$+ \left(1 - \frac{1}{\varepsilon}\right)(I - \varepsilon\mathcal{A})^{-1} f = \frac{1}{\varepsilon} f + \frac{\varepsilon - 1}{\varepsilon} f_\varepsilon \in Z.$$

Hence $f_\varepsilon = R(1) g_\varepsilon$, with $g_\varepsilon \in Z$ and it results that $f_\varepsilon \in D(\mathcal{A}_1)$, that is $\mathcal{D} \subset \overline{D(\mathcal{A}_1)}$.

Proof of Proposition 4. Firstly we observe that $\mathcal{B}_1$ is correctly defined. Because μ is a Lipschitz function with the constant L, it is sufficiently to demonstrate that for $\varphi \in \mathcal{D}$ it results that $\mathcal{B}_1 \varphi \in \mathcal{D}$.

Let $\varphi \in \mathcal{D}$, that is $\mid \varphi(x_1) - \varphi(x_2) \mid \leqslant L_1 \mid x_1 - x_2 \mid$, $\forall x_1, x_2 \in \overline{D(A)}$ and $\mid \varphi(x) \mid \leqslant M$, $\forall x \in \overline{D(A)}$. Then, for $\mathcal{B}_1 \varphi$ we have

$$\mid (\mathcal{B}_1 \varphi)(x_1) - (\mathcal{B}_1 \varphi)(x_2) \mid \leqslant L \mid \varphi(x_1) - \varphi(x_2) \mid \leqslant L L_1 \mid x_1 - x_2 \mid$$

and it results that $\mathcal{B}_1 \varphi \in \mathrm{LIP}\,(\overline{D(A)})$; then

$$\mid (\mathcal{B}_1 \varphi)(x) \mid \leqslant \mid (\mathcal{B}_1 \varphi)(x) - (\mathcal{B}_1 \varphi)(x_0) \mid + \mid (\mathcal{B}_1 \varphi)(x_0) \mid \leqslant$$

$$\leqslant L \mid \varphi(x) - \varphi(x_0) \mid + M_1 \leqslant 2LM + M_1,$$

where x_0 is a fixed element from $\overline{D(A)}$. Hence $\mathcal{B}_1 \varphi$ is bounded.

Moreover, if

$$\mid \varphi(y(t, x, u)) - \varphi(x) \mid \leqslant L_\varphi \left(t + \int_0^t \| u(x) \| ds \right), \quad \forall t \in [0, 1],$$

then

$$\mid (\mathcal{B}_1 \varphi)(y(t, x, u)) - (B_1 \varphi)(x) \mid \leqslant L L_\varphi \left(t + \int_0^t \| u(s) \| ds \right), \quad \forall t \in [0, 1].$$

Therefore $\mathcal{B}_1$ is correctly defined.

Since μ is a Lipschitz continuous function, it results that $\mathcal{B}_1$ is a continuous operator and the monotony of μ implies that $\mathcal{B}_1$ is dissipative. Indeed, for the second above affirmation, we have

$$\|\varphi_1 - \varphi_2\|_b \leqslant \|\varphi_1 - \varphi_2 - \lambda(\mathcal{B}_1 \varphi_1 - \mathcal{B}_1 \varphi_2)\|_b, \quad \forall \varphi_1, \varphi_2 \in \mathcal{D}, \quad \forall \lambda > 0. \tag{9}$$

The relation (9) results from the following inequality, that is valid for any $x \in \overline{\mathcal{D}(A)}$

$$| \varphi_1(x) - \varphi_2(x) | \leqslant | \varphi_1(x) - \varphi_2(x) + \lambda(\mu(\varphi_1(x)) - \mu(\varphi_2(x))) |.$$

Since $\mathcal{B}_1$ is a continuous operator, it results that it is dissipative on Z.

Proof of Theorem. In [1] it was demonstrated that for $g \in Y$

$$S(t)\varphi_0 = T(t)\varphi_0, \quad \forall \varphi_0 \in Z, \tag{10}$$

where $T(t) : \overline{D(\mathcal{A})} \to \overline{D(\mathcal{A})}$ is the semigroup generated by the operator $\mathcal{A}^0$, defined by the exponential formula

$$T(t)\varphi_0 = \lim_{n \to \infty} \left(I - \frac{t}{n} \mathcal{A}^0 \right)^{-n} \varphi_0, \quad \forall t \geqslant 0, \quad \varphi_0 \in Z.$$

Since $\mathcal{A}_1^0 = \mathcal{A}/_{D(\mathcal{A}_1)}$ it results that the semigroup $T(t)$ generated by $\mathcal{A}_1^0$ coincides with the semigroup generated by $\mathcal{A}^0$. For this reason we will use the same notation for both semigroups.

The equality (10) was demonstrated with the aid of the nonlinear version of the Chernoff theorem, namely it was showed that

$$\lim_{\varepsilon \searrow 0} \left(I - \lambda \frac{S(\varepsilon) - I}{\varepsilon} \right)^{-1} \varphi_0 = (I - \lambda \mathcal{A}^0)^{-1} \varphi_0, \quad \varphi_0 \in Z, \quad \lambda > 0, \tag{11}$$

with $S(t)$, $t \geqslant 0$ defined in (4).

Combining (10), (11) with the above observations, we obtain

$$\lim_{\varepsilon \searrow 0} \left(I - \lambda \frac{T(\varepsilon) - I}{\varepsilon} \right)^{-1} \varphi_0 = (I - \lambda \mathcal{A}_1^0)^{-1} \varphi_0, \quad \forall \varphi_0 \in Z, \quad \lambda > 0. \tag{12}$$

Now, the Theorem results from the Propositions 1...4 and from the relation (12).

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OBSERVAŢII ASUPRA UNEI ECUAŢII HAMILTON-JACOBI

(Rezumat)

Se demonstrează că soluţia ecuaţiei de evoluţie (6) asociată unei ecuaţii Hamilton-Jacobi, într-un spaţiu Hilbert, este dată de formula lui Lie-Trotter.

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ON n-HOMOGENEOUS POLYNOMINAL DIFFERENTIAL EQUATIONS

BY

ILIE BURDUJAN

0. Introduction. In this paper the n-homogeneous polynominal (autonomous) differential equations (briefly n-H.P.D.Eqs.) are analysed for $n \geqslant 3$. The case $n=2$ has been analysed, in the same manner, in [1]. The method used for this study generalizes the one used by L. Markus for quadratic differential systems [5]. To every such an equation a certain commutative n-algebra is associated. This correspondence becomes a 1 : 1 mapping between the classes of equivalent n-H.P.D.Eqs. (see Definition 1) and the classes of isomorphic commutative n-algebras (Theorem 1). The solution of Cauchy problem for such equations is an analytic function (Theorem 2). Moreover, if the n-algebra is a power-associative one, a formula for solving the Cauchy problem for such equations is given. At the same time, a practical procedure for solving the problem is derived, too. The last section of our paper analyses how the properties of the isomorphic commutative n-algebras are reflected in those ones of n-H.P.D.Eqs. which they are associated to.

1. Preliminaries. Throughout this paper the following notations will be used : E — a real Banach space ; $C(E)$ — the set of all the continuous functions from E to E ; $L(E^n, E)$ — the real Banach space of all the continuous n-linear simmetric forms from E^n to E, en dowed with the Banach space norm induced by the one of E ; Δ —the diagonal mapping from E to E^n (i.e., $\Delta(x)=(\underbrace{x,\ldots,x}_{n \text{ times}})$, $\forall x \in E$)).

The mapping $\mathbf{P} : L(E^n, E) \to C(E)$ defined by

$$\mathbf{P}(G)=G \circ \Delta, \quad \forall G \in L(E^n, E), \tag{1}$$

is called the *polynomial projection*. For every $G \in L(E^n, E)$, $\mathbf{P}(G)$ will be called the continuous *n-homogeneous polynomial* associated with G.

Let $P_n(E)$ denote the set of all the continuous n-homogeneous polynomials defined on E. Obviously, $P_n(E)=\mathbf{P}(L(E^n, E))$. The mapping $\mathbf{P} : L(E^n, E) \to P_n(E)$ (above defined by (1)) is a bijection because, for any $F \in P_n(E)$, the mapping $G \in L(E^n, E)$ defined by

$$G(x_1,\ldots,x_n)=\frac{1}{n!}\left[F\left(\sum_{i=1}^{n} x_i\right)-\sum_{j=1}^{n} F\left(\sum_{\substack{i=1\\ i\neq j}}^{n} x_i\right)+\sum_{\substack{j,k=1\\ j<k}}^{n} F\left(\sum_{\substack{i=1\\ i\notin\{j,k\}}}^{n} x_i\right)+\right.$$
$$\left.+\ldots+(-1)^{n-1}\sum_{i=1}^{n} F(x_i)\right], \quad \forall x_1,\ldots,x_n \in E, \tag{2}$$

is a n-linear symmetric form satisfying condition $F=G\circ\Delta$; this G will be called the *polar form* of the homogeneous polynomial F.

Any n-linear symmetric form $G\in L(E^n, E)$ endows the linear space E with a commutative n-ary operation defined by

$$[x_1,\dots, x_n]=G(x_1,\dots, x_n) \quad \forall x_1,\dots, x_n\in E. \tag{3}$$

Every element of the set $\{x_1,\dots, x_n\}$ appearing in the element $[x_1,\dots, x_n]$ will be called a factor of $[x_1,\dots, x_n]$. The n-algebra defined on E via the n-ary operation associated with G by (3) will be denoted either by $E(G)$, or by $E([,\dots,])$.

If $\|\ \|_1$ is the Banach space norm on E, let us define an equivalent Banach space norm $\|\ \| : E\to\mathbb{R}$ by

$$\|x\|=\sqrt[n-1]{\|G\|_1}\,\|x\|_1, \quad \forall x\in E. \tag{4}$$

The algebra $E(G)$, with E endowed with the norm $\|\ \|$, is called the B_n^*-algebra associated to the n-form G and will be denoted by $E(G, *)$.

The following sequence of sets $\{[x]^n\}_{n\in\mathbb{N}^*}$ is defined, by recurrence, for any $x\in E$:

(5) 1°. $[x]^1=\{x^{[1]}\}$ where $x^1=[\underbrace{x,\dots, x}_{n \text{ times}}]$;

2° $[x]^{k+1}$ is the set of all the elements obtained from the elements of $[x]^k$ by changing one and only one factor x by $x^{[1]}$.

An element $x\in E$ is called a *power-associative* element if card $[x]^k=1$ for every $k\in\mathbb{N}^*$; in this case, the unique element of $[x]^k$ will be denoted by $x^{[k]}$ and it is obtained by composing (via $[,\dots,]$) $k(n-1)+1$ elements identical to x.

The algebra $E(G)$ is called a *power-associative algebra* if every $x\in E$ is a power-associative element.

The following power series will be useful in the sequel

$$[1-(n-1)x]^{-\frac{1}{n-1}}=\sum_{i=0}^{\infty}\frac{\alpha_i}{i!}x^i, \tag{6}$$

where

$$\alpha_i=(-1)^i(n-1)^i\binom{-\frac{1}{n-1}}{i}. \tag{7}$$

This series converges for $|x|<1/(n-1)$.

For a linear continuous operator $L : E\to E$, let us define the mapping $T(\theta, L) : E\to E$, by

$$T(\theta, L)=\sum_{i=0}^{\infty}\frac{\alpha_i\theta^i}{i!}L^i, \tag{8}$$

where $L^i=\underbrace{L\circ\dots\circ L}_{i \text{ times}}(\forall i\in\mathbb{N}^*)$ and $L^0=I$ (I=identity operator on E). This mapping is defined for $|\theta|<1/(n-1)\|L\|$. Following the analogy with series(6), the notation

$$T(\theta, L)=(I-(n-1)\theta L)^{-\frac{1}{n-1}} \tag{9}$$

is considered.

2. In this section the n-homogeneous polynomial differential equations will be defined. For every such an equation, a commutative n-algebra can be associated in a natural way; moreover, this association is such that for any two equivalent n-H.P.D.Eqs. their associated n-algebras are isomorphic.

Let E be a real Banach space.

D e f i n i t i o n 1. Any autonomous differential equation of the form

$$\frac{dX}{dt}=F(X), \tag{10}$$

where $F\in P_n(E)$, is called a *n-homogeneous polynomial differential equation* (n-H.P.D.E.) on E.

It is well known [2] that there exists a unique solution of Eq. (10) satisfying

$$X(t_0)=X_0. \tag{11}$$

In what follows, the Cauchy problem (10)—(11) will be analysed for $n\geqslant 3$ only (the case $n=2$ has been analysed in [1] from the same point of view).

The homogeneous polynominal $F\in P_n(E)$ has the polar form G (defined by (2)), which organizes E as a commutative B_n^*-algebra as it has above been pointed out (via (3) and (4)).

Let now

$$\frac{dY}{dt}=F_1(Y) \tag{12}$$

be another n-H.P.D.E. with $F_1\in P_n(E)$.

D e f i n i t i o n 2. It is said that the Eqs. (10) and (12) are equivalent iff there exists a continuous automorphism of Banach spaces $h: E\to E$ such that $X=h(Y)$ is a solution of (10) whenever Y is a solution of (12).

Remark 1. Definition 2 introduces a binary relation on the set of all the n-H.P.D.Eqs. which is even an equivalence on this set, leading to a partition of it into mutually disjoint classes consisting of equivalent equations.

If the homogeneous polynomial F_1 has the polar form G_1 the commutative n-algebra $E(G_1)$ with the operation

$$[x_1,\dots, x_n]=G_1(x_1,\dots, x_n), \quad \forall x_1,\dots, x_n\in E, \tag{13}$$

may be defined on E.

T h e o r e m 1. *The n-H.P.D.Eqs.* (10) *and* (12) *are equivalent iff their associated n-algebras* $E(G)$ *and* $E(G_1)$ *are isomorphic.*

P r o o f. The condition for (10) and (12) to be equivalent implies

$$h(F_1(Y))=F(h(Y)), \tag{14}$$

for any solution Y of (12). In particular, for $t=t_0$, (14) implies that $h(F_1(y_0))=$ $=F(h(y_0))$, where $y_0=Y(t_0)$. But, for every arbitrarily chosen (but fixed) $u\in E$, the Cauchy problem for Eq. (12) has a unique solution. It follows that

$$h(F_1(u))=F(h(u)), \quad \forall u\in E. \tag{15}$$

From (2), (3), (13) and (15) it results that h is an isomorphism between n-algebras $E(G)$ and $E(G_1)$. Conversely, let us assume that the two n-algebras $E(G)$ and $E(G_1)$ are isomorphic to each other. Then there exists a Banach space automorphism $h: E \to E$ such that $h([u_1,\dots, u_n]) = [h(u_1),\dots, h(u_n)]$, for any $u_1,\dots, u_n \in E$. For any solution $Y: I \subset \mathbb{R} \to E$ of (12) one gets $h(F_1(Y)) = F(h(Y))$. The last equality allows us to prove that $X = h(Y)$ is a solution of Eq. (10).

Remark 2. Based on Theorem 1, a 1 : 1 mapping between the classes of equivalent n-H.P.D.Eqs. and the classes of isomorphic commutative n-algebras is built up. Consequently, the properties of commutative n-algebras invariant under isomorphisms, correspond with the properties of n-H.P.D.Eqs. Unfortunately, such a king of study cannot get to far because the n-algebras were insufficiently studied.

3. In this section it will be shown that the solution of the Cauchy problem (10)—(11) is analytic everywhere it exists.

Let us consider the B_n^*-algebra $E(G)$ associated to the Eq. (10) and let us denote by $X: I \subset \mathbb{R} \to E\ (t \in \operatorname{Int} I)$ the solution of Cauchy problem (10)—(11). Since $F(X) = G(X,\dots,X)$ is an indefinitely differentiable function, then dX/dt is indefinitely differentiable on I, too. For the solution X of Cauchy problem (10)—(11), the following equations can be derived :

$$\begin{cases} \dfrac{dX}{dt} = X^{[1]}, \\[2ex] \dfrac{d^2X}{dt^2} = nX^{[2]}, \\[2ex] \dfrac{d^3X}{dt^3} = n^2[X^{[2]}, X,\dots,X] + n(n-1)[X^{[1]}, X^{[1]}, X,\dots, X]. \end{cases}$$

Let us remark that the third derivative has α_3 terms with $3(n-1)+1$ factors. Then, the following inductive assumption can be stated and proved : the i-th derivative of X has α_i terms, everyone of them having $i(n-1)+1$ factors.

Since $E(G)$ is a B_n^*-algebra, then

$$\left\| \frac{d^iX}{dt^i} \right\| \leqslant \alpha_i \|X\|^{i(n-1)+1}, \quad \forall i \in \mathbb{N}^*,\ \forall t \in I. \tag{16}$$

In particular, for a fixed $\tau \in I$,

$$\alpha_0\|X(\tau)\| + \frac{\alpha_1}{1!}|t-\tau|\,\|X(\tau)\|^n + \frac{\alpha_2}{2!}|t-\tau|^2\|X(\tau)\|^{2n-1} + \dots + \tag{17}$$

$$+ \frac{\alpha_i}{i!}|t-\tau|^i\|X(\tau)\|^{i(n-1)+1} + \dots = \|X(\tau)\|[1-(n-1)|t-\tau|\,\|X(\tau)\|^{n-1}]^{-\frac{1}{n-1}}$$

holds for $(n-1)|t-\tau|\,\|X(\tau)\|^{n-1} < 1$. Thus, the Taylor expansion for $X(t)$ is absolutely and uniformly convergent for $(n-1)\|X\|^{n-1}|t-\tau| < 1$. Now, let $J \subset I$ be a compact interval. For any $t \in J$, one gets from (16) the inequality

$$\left\| \frac{d^i X}{dt^i} \right\| \leqslant \alpha_i (\sup_{t \in J} \| X(t) \|)^{i(n-1)+1}, \quad \forall i \in \mathbb{N}^*. \tag{18}$$

Using the notations

$$K_J = \sup_{t \in J} \| X(t) \|, \quad \delta_J = \frac{1}{n K_J^{n-1}}, \tag{19}$$

it can be derived from (18) that

$$\frac{1}{i!} \left\| \frac{d^i X}{dt^i} \right\| \delta_J^i < K_J.$$

The result is that, the Taylor expansion of $X(t)$ converges on J just to $X(t)$ (see [3], p. 307). Since for every $t \in I$ there exists a compact interval $J \subset I$ such that $t \in J$, the following result has been obtained :

T h e o r e m 2. *The solution of the Cauchy problem* (10) —(11) *is an analytic mapping on whole its domain of existence.*

4. In this section the analytic solution of Cauchy problem (10) —(11) will be effectively determined under the assumption that the associated B_n^*-algebra is a power-associative one.

Throughout this section the n-algebra $E(G)$ will be considered a power-associative algebra.

Let us first define the linear operator $G_{X_0} : E \to E$ by

$$G_{X_0}(x) = [\underbrace{X_0, X_0, \ldots, X_0}_{n-1 \text{ times}}, x], \quad \forall x \in E. \tag{20}$$

Then equalities

$$\left. \frac{dX^i}{dt^i} \right|_{t=t_0} = \alpha_i G_{X_0}^i (X_0) \quad \forall i \in \mathbb{N}^* \tag{21}$$

hold (in which $G_{X_0}^i = \underbrace{G_{X_0} \circ \ldots \circ G_{X_0}}_{i \text{ times}}$). The Taylor expansion of solution $X(t)$ will be

$$X(t) = \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} (t - t_0)^i G_{X_0}^i (X_0), \tag{22}$$

i.e., using (9),

$$X(t) = [I - (n-1)(t - t_0) G_{X_0}]^{-\frac{1}{n-1}} (X_0) \tag{23}$$

holds. It is thus proved

T h e o r e m 3. *If the B_n^*-algebra $E(G)$ associated to Eq.* (10) *is a power-associative one, the solution of Cauchy problem* (10) —(11) *is given by* (23), *for* $|t - t_0| < 1/(n-1)\|G_{X_0}\|$.

Remark 3. The p r o o f of Theorem 3 effectively used the associativity of the powers of X_0, only. Consequently the formula (23) can be used even when the algebra $E(G)$ is not a power-associativ one, but X_0 has associative powers.

Let us now consider the particular case when $E = \mathbb{R}^p$. By use of a basis $B = \{e_1, \ldots, e_p\} \subset E$, the Eq. (10) becomes

$$\frac{dX^i}{dt} = C^i_{i_1 \ldots i_n} X^{i_1} \ldots X^{i_n}, \quad i_1, \ldots, i_n = 1, 2, \ldots, p, \tag{24}$$

where $X(t)=\sum_{i=1}^{p} X^i e_i$; condition (11) also turns to

$$X^i(t_0)=X_0^i \quad i=1, 2,\dots, p. \tag{25}$$

Here $C^i_{i_1\dots i_n}$ denote the structure constants of $E(G)$. Then, the linear operator G_{X_0} defined by (20) has the matrix

$$[G_{X_0}]_B=[C^i_{i_1\dots i_{n-1}} X_0^{i_1}\dots X_0^{i_{n-1}}]$$

in the basis B. Let us represent the vector space E as direct sum $E=E_1\oplus E_2$, where E_1 and E_2 are the invariant subspaces of G_{X_0} corresponding to the real eigenvalues and to the complex eigenvalues, respectively. Then, there exists the basis B_1 and B_2 in E_1 and E_2, respectively, such that $B_c=B_1\cup B_2$ is a basis in E, in which the corresponding matrix of G_{X_0} is of the form

$$G_{X_0B_c}=\begin{bmatrix} J_1 & & & & 0 \\ & \ddots & & & \\ & & J_s & & \\ & & & R_1 & \\ 0 & & & & \ddots \; R_q \end{bmatrix}. \tag{26}$$

Hence it is a diagonal block matrix with superior Jordan blocks $J_i (i=1,2,\dots,n)$ and the blocks matrix

$$R_j=\begin{bmatrix} S_j & I_2 & 0 & \dots & 0 \\ 0 & S_j & I_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & I_2 \\ 0 & 0 & 0 & \dots & S_j \end{bmatrix}_{m_j} \quad \text{with} \quad S_j=\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}, \quad I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{27}$$

if $\lambda_j=a_j+ib_j$ is an eigenvalue of G_{X_0} with the multiplicity m_j.

If A is the transformation matrix from basis B to basis B_c, then

$$G_{X_0B_c}=A^{-1}[G_{X_0}]_B A \tag{28}$$

holds. If

$$J_i=\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & & 0 & \lambda_i \end{bmatrix}_{m_i\times m_i} \tag{29}$$

and

$$f(\lambda)=[1-(n-1)(t-t_0)\lambda]^{-\frac{1}{n-1}} \tag{30}$$

is an analytic function in a neighborhood of $\lambda=\lambda_i$, the (see [4], p. 17)

$$f(J_i)=\begin{bmatrix} f(\lambda_i) & \frac{1}{1!}f'(\lambda_i) & \frac{1}{2!}f''(\lambda_i) & \dots & \frac{1}{(m_i-1)!}f^{(m_i-1)}(\lambda_i) \\ 0 & f(\lambda_i) & \frac{1}{1!}f'(\lambda_i) & \dots & \frac{1}{(m_i-2)!}f^{(m_i-2)}(\lambda_i) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{1!}f'(\lambda_i) \\ 0 & 0 & 0 & \dots & f(\lambda_i) \end{bmatrix}_{m_i\times m_i} \tag{31}$$

holds. On the other hand, one obtains

$$(32)\qquad f(R_j)=\begin{bmatrix} f(S_j) & \frac{1}{1!}f'(S_j) & \frac{1}{2!}f''(S_j) & \dots & \frac{1}{(m_j-1)!}f^{(m_j-1)}(S_j) \\ 0 & f(S_j) & \frac{1}{1!}f'(S_j) & \dots & \frac{1}{(m_j-2)!}f^{(m_j-2)}(S_j) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{1!}f'(S_j) \\ 0 & 0 & 0 & \dots & f(S_j) \end{bmatrix}.$$

For $\lambda_j=\alpha_j+ib_j=r_j(\cos\theta_j+i\sin\theta_j)$

$$(33)\qquad S_j=r_j\begin{bmatrix}\cos\theta_j & \sin\theta_j\\ -\sin\theta_j & \cos\theta_j\end{bmatrix}\quad\text{and}\quad S_j^k=r_j^k\begin{bmatrix}\cos k\theta_j & \sin k\theta_j\\ -\sin k\theta_j & \cos k\theta_j\end{bmatrix}$$

holds. Consequently, one derives

$$(34)\qquad f(S_j)=\begin{bmatrix}A_j & B_j\\ -B_j & A_j\end{bmatrix}$$

with

$$(35)\qquad A_j=\sum_{k=0}^{\infty}\frac{\alpha_k}{k!}r_j^k\cos k\theta_j,\qquad B_j=\sum_{k=0}^{\infty}\frac{\alpha_k}{k!}r_j^k\sin k\theta_j.$$

From (35) one obtains

$$(36)\qquad A_j+i\,B_j=[1-(n-1)r_j(t-t_0)(\cos\theta_j+i\sin\theta_j)]^{-\frac{1}{n-1}}.$$

Here is considered that

$$(37)\qquad \begin{aligned}&[1-(n-1)(t-t_0)r_j(t-t_0)(\cos\theta_j+i\sin\theta_j)]^{-\frac{1}{n-1}}=\\ &=[1+(n-1)^2(t-t_0)^2r_j^2-2(n-1)(t-t_0)r_j\cos\theta_j]^{-\frac{1}{n-1}}\left(\cos\frac{\omega_j}{n-1}-i\sin\frac{\omega_j}{n-1}\right),\end{aligned}$$

with

$$(38)\qquad \omega_j=\arg[1-(n-1)(t-t_0)r_j(\cos\theta_j+i\sin\theta_j)],\quad \omega_j\in[0,2\pi).$$

Therefore we have

$$(39)\qquad \begin{aligned}A_j&=[1+(n-1)^2(t-t_0)^2r_j^2-2(n-1)(t-t_0)r_j\cos\theta_j]^{-\frac{1}{n-1}}\cos\frac{\omega_j}{n-1},\\ B_j&=-[1+(n-1)^2(t-t_0)^2r_j^2-2(n-1)(t-t_0)r_j\cos\theta_j]^{-\frac{1}{n-1}}\sin\frac{\omega_j}{n-1}.\end{aligned}$$

It has thus been obtained

$$(40)\qquad f([G_{X_0}]_B)=A\begin{bmatrix} f(J_1) & & & & & 0\\ & \ddots & & & & \\ & & f(J_s) & & & \\ & & & f(R_1) & & \\ & & & & \ddots & \\ 0 & & & & & f(R_q)\end{bmatrix}A^{-1}.$$

Hence, the solution of problem (10)—(11) is

$$(41)\qquad [X(t)]_B=f([G_{X_0}]_B)\,[X_0]_B.$$

5. In this section it will be seen how some properties of the n-algebra $E(G)$ (invarianted under isomorphisms) are reflected in the properties of Eq. (10). Unfortunately, unlike the case of quadratic differential equations, this

problem is much harder to be approached in this case since the n-algebras have not yet been studied to a larger extent. However, the most part of the results established for the quadratic differential equations may be generalized for this case, too.

1. An element $X_0 \neq 0$ of E having the property

$$G_{X_0}(X_0)=0$$

will be called a first order nilpotent element of $E(G)$.

D e f i n i t i o n 3. The element $X_0 \in E$ is called a *critical point* of n-H.P.D.E. (10) if the constant function $X: \mathbb{R} \to E$, defined by $X(t) = X_0$ $(\forall t \in \mathbb{R})$ is a solution of (10).

Obviously, any first order nilpotent element of $E(G)$ is a critical point for (10). Moreover, $X=0$ is always a critical point for it. If $X_0 \in E \setminus \{0\}$ is a first order nilpotent element, any element $X=\lambda X_0$ (for $\lambda \in \mathbb{R}^*$) is a first order nilpotent element, too. Therefore, if (10) has a critical point $X_0 \neq 0$, then the straightline $X=\lambda X_0$ consists from critical points only. Thus, the element $X_0=0$ is an insolated critical point for (10) if and only if the algebra $E(G)$ has no first order nilpotent elements.

2. Now, let us deal with the implications of the existence of a first order idempotent element of $E(G)$. Recall that $X_0 \in E \setminus \{0\}$ is a first order idempotent element if $G_{X_0}(X_0)=X_0$. Since X_0 has associative powers, we seek the solution of Cauchy problem (10)—(11) of the form (in according to Theorem 3) $X(t)=f(t)X_0$. We obtain

$$X(t)=\frac{1}{\sqrt[n-1]{1-(n-1)(t-t_0)}}X_0.$$

Moreover, the elements $X_0 \in E^*$ for which $G_{X_0}(X_0)=\lambda X_0$ $(\lambda \in \mathbb{R}^*)$ have associative powers $(G^n_{X_0}(X_0)=\lambda^n X_0)$ and, from Theorem 3, one obtains

$$X(t)=\frac{1}{\sqrt[n-1]{1-(n-1)(t-t_0)\lambda}}X_0$$

as the solution of Cauchy problem (10)—(11).

3. The cases above analysed correspond to the situation when $E(G)$ has a 1-dimensional subalgebra for which the element $X(t_0)=X_0$ is a basis. Let us assume that $E(G)$ has a k-dimensional subalgebra E_1 containing $X_0 = X(t_0)$. Then there exists a Banach subspace E_2 such that $E=E_1 \oplus E_2$, as well as the natural projections $p_i: E \to E_i$. Then, every power of X_0 may be expressed as a linear combination of the elements of a certain basis $B=\{e_1,\dots,e_k\} \subset E_1$, so that the solution of (10)—(11) is of the form $X(t)=f^1(t)e_1+\dots+f^k(t)e_k$. The condition $\dot{X}(t)=F(X(t))$ becomes $\dot{f}^i=C^i_{j_1\dots j_h} f^{j_1}\dots f^{j_h}$ $(i=1, 2,\dots, k)$, with the condition $f^s(t_0)=X^s_0$ (when $X_0=\sum_{s=1}^{k} X^s_0 e_s$). Here $C^s_{j_1\dots j_h}$ are the structure constants of subalgebra $E_1(G)$ relative to basis B. Therefore, the Cauchy problem for the given n-H.P.D.E. is transformed in a Cauchy problem for a polynomial system.

4. Let now assume that the linear subspace $E_1 \subset E$ is a proper ideal of $E(G)$ (i.e., $G(E_1, E, \ldots, E) \subset E_1$), closed in E. If $E = E_1 \oplus E_2$ and $p_i : E \to E_i$ $(i = 1, 2)$ are the projections associated with this direct sum, then any solution $X(t)$ of (10) may be uniquely written as $X(t) = X_1(t) + X_2(t)$ and the Eq. (10) becomes

$$(42) \qquad \begin{cases} \dfrac{dX_1}{dt} = \sum\limits_{k=0}^{k-1} C_n^k G(X_1, \underbrace{X_2 \ldots, X_2}_{k \text{ times}}) + p_1 G(X_2, \ldots, X_2), \\ \dfrac{dX_2}{dt} = p_2 G(X_2, \ldots, X_2). \end{cases}$$

Moreover, if E_2 is subalgebra of $E(G)$, then $p_1 G(X_2, \ldots, X_2) = 0$ and $p_2 G(X_2, \ldots, X_2) = G(X_2, \ldots, X_2)$. If E_2 is just an ideal then (42) becomes

$$(43) \qquad \frac{dX_1}{dt} = G(X_1, \ldots, X_1), \qquad \frac{dX_2}{dt} = G(X_2, \ldots, X_2).$$

5. An element $X_0 \in E^*$ is called a vanishing element of $E(G)$ if $G(X_0, X_2 \ldots, X_n) = 0$ for any $X_2, \ldots, X_n \in E$. If $E_1 = \{X \in E / G(X, X_2, \ldots, X_n) = 0, \forall X_2, \ldots, X_n \in E\}$, then E_1 is an ideal of $E(G)$ closed in E. If $E = E_1 \oplus E_2$, then one obtains

$$(44) \qquad \frac{dX_1}{dt} = p_1 G(X_2, \ldots, X_2), \qquad \frac{dX_2}{dt} = p_2 G(X_2, \ldots, X_2).$$

6. Let $E(G)$ be the n-algebra associated with the Eq. (10). If $E(G)$ is a nilalgebra, then the solution of Cauchy problem (10)—(11) is a polynominal function; while if $E(G)$ is either a semisimple algebra or a solvable one, the integral line of (10) passing through X_0 (at $t = t_0$) lies in a hyperplane of E.

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ASUPRA ECUAŢIILOR DIFERENŢIALE POLINOMIALE n-OMOGENE

(Rezumat)

Se consideră ecuaţiile diferenţiale polinominale n-omogene, cu $n > 3$, în spaţii Banach. Fiecărei astfel de ecuaţii i se asociază o anumită n-algebră comutativă. Dacă această algebră este cu puteri asociative, se determină o formulă de rezolvare a problemei Cauchy. În plus, se dă o procedură practică de rezolvare a acestor ecuaţii în cazul în care spaţiul Banach pe care sînt definite este de dimensiune finită.

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ACCELERATION WAVES IN LINEAR THERMOELASTIC MATERIALS

BY

I. NISTOR

1. Basic Equations and Jump Conditions. A fixed system of rectangular cartesian coordinates and cartesian tensor notation are used throughout. We denote by u_i the components of displacement vector field and a superposed dot denotes the material time derivative. Also ρ is mass density at time t and ρ_0 is the reference mass density.

We assume that in present configuration the body is acted upon by a system of externally applied forces consisting of an external body force represented by a vector field F_i per unit mass and an external surface force over the bounding of the body specified by a vector $t_{(n)i}$ per unit area in present configuration. We write the equation of balance of linear momentum for material volume $\mathcal{P}$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \rho \dot{u}_i \mathrm{d}v = \int_{\mathcal{P}} \rho F_i \mathrm{d}v + \int_{\partial\mathcal{P}} t_{(n)i} \mathrm{d}a, \tag{1.1}$$

where $\mathrm{d}v$ and $\mathrm{d}a$ are elements of volume and area in the present configuration.

The equation of balance of energy for each material volume $\mathcal{P}$ in the present configuration may be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \rho\left(\frac{\dot{u}_i\dot{u}_i}{2} + \varepsilon\right)\mathrm{d}v = \int_{\mathcal{P}} \rho(r + F_i\dot{u}_i)\mathrm{d}v + \int_{\partial\mathcal{P}} (t_{(n)i}\dot{u}_i - h_{(n)})\mathrm{d}a, \tag{1.2}$$

where ε is the internal energy per unit mass, $h_{(n)}$ is the heat flux across the surface $\partial\mathcal{P}$, r is the heat supply per unit mass and per unit time, and $n=(n_i)$ is the unit outward normal to the surface $\partial\mathcal{P}$.

Green and Laws [2] proposed (for homogeneous bodies) an entropy production inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \rho\eta \,\mathrm{d}v - \int_{\mathcal{P}} \frac{\rho r}{\Phi}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \frac{h_{(n)}}{\Phi}\,\mathrm{d}a \geqslant 0, \tag{1.3}$$

for every material volume $\mathcal{P}$, where η is the specific entropy, Φ ($\Phi>0$) is a scalar function.

It is well known that $t_{(n)i}$ and $h_{(n)}$, as functions of n, satisfy the following equations (see [3], pp. 28, 37)

$$t_{(n)i} = -t_{(-n)i}, \quad h_{(n)} = -h_{(-n)}. \tag{1.4}$$

Green and Lindsay [1] have given a generalized theory of classical thermoelasticity using inequality (1.3). Here we consider the linear theory of thermoelasticity established by Green and Lindsay [1].

We use x_i to denote points in the reference configuration of the body and $,_i$ to denote partial differentiation with respect to x_i. The components of linear strain tensor are

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.5}$$

Assuming that the initial body is free from stress and has zero heat flux [1] we have

$$\begin{gathered} t_{ij} = k_{ijrs}e_{rs} + a_{ij}(\theta + \alpha\dot{\theta}), \\ q_i = -\theta_0(b_i\dot{\theta} + k_{ij}\theta_{,j}), \\ \rho_0\eta = a + d\theta + h\dot{\theta} - a_{ij}e_{ij} - b_i\theta_{,i}, \\ \sigma = \sigma_0 - a(\theta + \alpha\dot{\theta}) - e\theta\dot{\theta} + \alpha b_i\theta\theta_{,i} - \tfrac{1}{2}d\theta^2 - \tfrac{1}{2}f\dot{\theta}^2 + \\ + a_{ij}e_{ij}(\theta + \alpha\dot{\theta}) + \tfrac{1}{2}(k_{ijrs}e_{ij}e_{rs} + \alpha k_{ij}\theta_{,i}\theta_{,j}), \\ \Phi = \theta_0 + \theta + \alpha\dot{\theta} + \beta\theta\dot{\theta} + \tfrac{1}{2}\gamma\dot{\theta}^2, \end{gathered} \tag{1.6}$$

where

$$b = a\alpha, \quad e = a\beta + d\alpha, \quad \alpha h = f - \frac{b\gamma}{\alpha}, \tag{1.7}$$

$$a_{ij} = a_{ji}, \quad k_{ij} = k_{ji}, \quad k_{ijrs} = k_{rsij}, \quad k_{ijrs} = k_{jirs} = k_{ijsr}. \tag{1.8}$$

In the Eqs. (1.6) t_{ij} are the components of the stress tensor, θ is the temperature measured from a constant reference temperature θ_0, σ is the Helmholtz free energy function defined by

$$\sigma = \varepsilon - \eta\Phi.$$

From the local form of the inequality (1.3) we have

$$\alpha d - h \geqslant 0, \tag{1.9}$$

$$k_{ij}\lambda_i\lambda_j \geqslant 0, \tag{1.10}$$

for every vector (λ_i).

In the regions where t_{ij}, u_i, ε and q_i are continuously differentiable, while $\ddot{u}_i$, F_i, r, $\dot{\varepsilon}$ are continuous, Eqs. (1.1) and (1.2) yields [2] the local form

$$\begin{gathered} t_{ij,i} + \rho_0 F_j = \rho_0\ddot{u}_j, \\ k_{ij}\theta_{,ij} - d\dot{\theta} - h\ddot{\theta} + a_{ij}\dot{e}_{ij} + 2b_i\dot{\theta}_{,i} + \frac{\rho_0 r}{\theta_0} = 0. \end{gathered} \tag{1.11}$$

The sufficient conditions for uniqueness of solution of the initial, mixed boundary-value problem are

$$h>0, \quad k_{ijrs}e_{ij}e_{rs}\geqslant 0, \quad \theta_0>0, \quad \alpha>0, \quad \rho_0>0. \tag{1.12}$$

A surface $\Sigma(x_j, t)$ that is singular with respect to some quantity and that has a nonzero speed of propagation is said to be a propagating singular surface or wave.

Acceleration waves are propagating singular surface on which the quantities $u_{i,j}$, $\dot{u}_i$, θ and η are continuous and some at least of their first partial derivatives discontinuous.

The geometrical and kinematical conditions of compatibility for acceleration waves [4] are

$$[u_{i,jk}]=V^{-2}a_i n_j n_k, \quad [\dot{u}_{i,j}]=-V^{-1}a_i n_j, \quad [\theta_{,i}]=gn_i, \quad [\dot{\theta}]=-Vg, \tag{1.13}$$

where

$$a_i=[\ddot{u}_i], \quad g=[\theta_{,p}n_p]. \tag{1.14}$$

Here n_i are the components of the unit normal vector in the direction of propagation of Σ, square brackets denote the jump in the enclosed quantity as Σ is crossed in the direction $(-n_i)$ and V is the speed of propagation.

We shall be concerned mainly with waves propagating into material which is homogeneously deformed, at rest and at uniform and constant temperature. When these conditions prevail we refer to the part of the body ahead of Σ as a region of uniform state. It follows from Eqs. (1.6) that the stress tensor, the heat flux vector, the Helmholtz free energy function and the function Φ are discontinuous on acceleration wave.

At an acceleration wave advancing into a region of uniform state we obtain from (1.1), (1.2) and (1.3) the following jump conditions

$$\begin{gathered}[\eta]=0, \quad [t_{ij}n_j]=0, \quad V[\sigma+\rho_0\eta\Phi]-[q_i n_i]=0, \\ [\rho_0\eta]-\left[\frac{q_i n_i}{\Phi}\right]\geqslant 0.\end{gathered} \tag{1.15}$$

From the relations (1.15) with use of the relations (1.5), (1.6) and (1.13) we find that

$$-Vgh-gb_i n_i=0, \quad g(V^2h+k_{ij}n_i n_j)\left(\tfrac{1}{2}Vg\alpha+\theta_0\right)=0, \quad g(V^2h+k_{ij}n_i n_j)\geqslant 0. \tag{1.16}$$

It follows from the relations (1.10) and (1.12) that

$$hV^2+k_{ij}n_i n_j>0. \tag{1.17}$$

An acceleration wave of zero thermal amplitude ($g=0$) is called an isothermal wave.

The basic result which we now establish is

T h e o r e m 1. *If the speed of propagation of an acceleration wave advancing into a region of uniform state is positive, then the wave is necessarily isothermal.*

The p r o o f of Theorem 1 is based upon the discontinuity relations (1.16) and the relation (1.17). Suppose that the thermal amplitude g is non-zero. From $(1.16)_2$ we obtain

$$g = -\frac{2\theta_0}{\alpha V} < 0,$$

and therefore the condition $(1.16)_3$ cannot be satisfied.

It follows that Σ is an isothermal wave.

2. Amplitudes of Acceleration Waves. After we substitute the relations (1.5), $(1.6)_1$ into the system of Eqs. (1.11) and use relations (1.8) we obtain

$$\begin{aligned} &\rho\ddot{u}_j - k_{ijrs}u_{r,si} - \alpha a_{ij}\dot{\theta}_{,i} - a_{ij}\theta_{,i} - \rho_0 F_j = 0, \\ &h\ddot{\theta} - 2b_i\dot{\theta}_{,i} - k_{ij}\theta_{,ij} - a_{ij}\dot{u}_{i,j} + d\dot{\theta} - \frac{r\rho_0}{\theta_0} = 0. \end{aligned} \tag{2.1}$$

Further we shall consider acceleration waves which are isothermal. In this case we have the following compatibility conditions

$$\begin{aligned} &[\theta_{,ij}] = V^{-2}a_4, \quad [\dot{\theta}_{,i}] = -V^{-1}a_4, \quad a_4 = [\ddot{\theta}], \\ &[\dot{u}_{i,j}] = a_{i,j} - [\ddot{u}_i]n_j V^{-1}, \quad [\dot{\theta}_{,j}] = a_{4,j} - [\ddot{\theta}]n_i V^{-1}, \\ &[\dot{u}_{i,jk}] = [\ddot{u}_i]n_j n_k V^{-2} - (a_i n_j V^{-1})_{,k} - a_{i,j}n_k V^{-1}, \\ &[\dot{\theta}_{,ij}] = [\ddot{\theta}]n_i n_j V^{-2} - (a_4 n_i V^{-1})_{,j} - a_{4,i}n_j V^{-1}. \end{aligned} \tag{2.2}$$

The differential equation (2.1) must hold on each side of the surface $\Sigma(x_j, t)$. On forming the jump of each term in Eqs. (2.1) on the singular surface and using the compatibility conditions (1.13) and (2.2) we obtain

$$\begin{aligned} &\rho_0 V^2 a_i - k_{ijrs}n_j n_r a_s + \alpha V a_{ij}n_j a_4 = 0, \\ &(V^2 h + 2b_i n_i V - k_{ij}n_i n_j)a_4 + V a_{ij}n_j a_i = 0. \end{aligned} \tag{2.3}$$

The vector $A = (a_k)$, $(k = 1, 2, 3, 4)$ is called the amplitude vector of the isothermal acceleration waves.

With the definitions

$$\begin{aligned} &B_{ij} = V^2\rho_0\delta_{ij} - K_{ij}, \quad B_{i4} = a_{ip}n_p V\alpha, \\ &B_{4j} = V a_{jq}n_q, \quad B_{44} = V^2 h + 2b_i n_i V - k_{ij}n_i n_j, \\ &K_{ij} = k_{irsj}n_r n_s, \end{aligned} \tag{2.4}$$

we can write (2.3) as

$$B_{ij}a_j = 0, \quad (i, j = 1, 2, 3, 4). \tag{2.5}$$

This set of four linear homogeneous equations for four unknown functions a_i $(i = 1, 2, 3, 4)$ admits a non-trivial solution if, and only if

$$\det(B_{ij}) = 0. \tag{2.6}$$

We see that Eq. (2.6) is an algebraic equation of degree eight in V.

P r o p o s i t i o n 1. *We note that, in particular case when is satisfied the condition*

$$b_i n_i = 0,$$

the Eq. (2.6) *is a polynominal equation of degree four in* V^2.

With $V=0$, it follows from (2.4) that

$$\det(B_{rs}) = -k_{ij}n_i n_j \det(K_{pq}). \tag{2.7}$$

We assume that

$$k_{ij}\lambda_i\lambda_j > 0, \quad k_{ijrs}\lambda_i\lambda_j\mu_r\mu_s > 0, \tag{2.8}$$

for all non-vanishing vectors (λ_i) and (μ_j).

P r o p o s i t i o n 2. *The Eq.* (2.6) *admits a non-trivial solution if, and only if, are satisfied the conditions* (2.8).

Proposition 2 follows immediately from (2.7).

Further, (2.3), in view of (2.8), is found to be equivalent to

$$\begin{gathered} K_{ij}(Va_j - a_{j+4}) = 0, \\ \alpha k_{ij}n_i n_j(Va_4 - a_8) = 0, \\ \rho_0 Va_{i+4} - K_{ij}a_j + \alpha a_{ij}n_j a_8 = 0, \\ \alpha(hV + 2b_i n_i)a_8 - k_{ij}n_i n_j a_4 \alpha + \alpha a_{ij}n_j a_{i+4} = 0, \end{gathered} \tag{2.9}$$

where

$$a_{i+4} = Va_i, \quad (i=1, 2, 3, 4).$$

We can write (2.9) in the form

$$(VM^0_{ij} + M_{ij})a_j = 0, \quad (i, j = 1, 2, \ldots, 8),$$

where the matrix (M^0_{ij}) is positive definite and symmetric while the matrix (M_{ij}) is symmetric.

The possible wave speeds are given by the solutions of the equation

$$\det(VM^0_{ij} + M_{ij}) = 0. \tag{2.10}$$

We easily find from (2.9) that

$$\det(VM^0_{ij} + M_{ij}) = \alpha k_{ij}n_i n_j \det(K_{pq}) \det(B_{rs}). \tag{2.11}$$

P r o p o s i t i o n 3. *If are satisfied the conditions* (2.8), *then all roots of the Eq.* (2.6) *must be real.*

P r o o f. Since (M^0_{ij}) is symmetric and positive definite and (M_{ij}) is symmetric ([4], p. 829) the roots of the Eq. (2.10) must be real. It follows from (2.11) that all the roots of the Eq. (2.6) are real. By Proposition 2 the roots of the Eq. (2.6) are non-trivial and hence by Proposition 1 we conclude that Eq. (2.6) admits positive roots. Thus we have

T h e o r e m 2. *The isothermal acceleration waves may propagate at any point of the medium and in any given direction provided that* (2.8) *holds at that point.*

If root V of the Eq. (2.6) has multiplicity β ($\beta \leqslant 4$) and if (f_{pi}), $(p=1, 2, \ldots \beta)$, are any β linearly independent solutions of the system (2.3) corresponding to the speed V, then the amplitude vector of acceleration wave can be expressed as

$$a_i = \sum_{p=1}^{\beta} \Psi_p f_{pi}. \tag{2.12}$$

Without loss of generality we can choose (f_{pi}) so that

$$\rho_0 V^2 f_{pi} f_{qi} + k_{ijrs} n_j n_s f_{pi} f_{qr} + \alpha(hV^2 + k_{ij} n_i n_j) f_{p4} f_{q4} = \delta_{pq}. \tag{2.13}$$

Since (f_{pi}) is a solution of the system (2.3) are satisfied the following relations

$$\begin{aligned} &k_{ijrs} n_i n_s f_{pr} - \alpha V a_{ij} n_i f_{p4} - \rho_0 V^2 f_{pj} = 0, \\ &-\alpha V a_{ij} n_j f_{pi} + \alpha(k_{ij} n_i n_j - 2b_i n_i h - hV^2) f_{p4} = 0. \end{aligned} \tag{2.14}$$

By differentiating (2.14) with respect to n_r at constant x_i and t, we get

$$\begin{aligned} &(k_{rjmi} + k_{ijmr}) n_i f_{pm} - \alpha V a_{jr} f_{p4} - (\alpha a_{ij} n_i f_{p4} + 2V f_{pj} \rho_0) V_{,n_i} = 0, \\ &-\alpha(V a_{ir} f_{pi} - 2k_{ri} n_i + 2b_r V) - \alpha\{a_{ij} n_j f_{pi} + \\ &\quad + 2(b_i n_i + Vh f_{p4})\} V_{,n_r} = 0. \end{aligned} \tag{2.15}$$

If, now, we multiply (2.15) by (f_{qj}), $(j=1, 2, 3, 4)$ and add with respect to j we find

$$\begin{aligned} V_{,n_r} V^{-1} \delta_{pq} = &(k_{rjmi} + k_{ijmr}) n_i f_{pm} f_{qj} - \alpha V a_{ri} (f_{pi} f_{q4} + f_{p4} f_{qi}) + \\ &+ 2\alpha(k_{ri} n_i - V b_r) f_{p4} f_{q4}. \end{aligned} \tag{2.16}$$

In deriving (2.16) we make use of the relations (2.13) and (2.14).

Our next objective is to derive a system of differential equations for Ψ_p. By taking the material derivative of the system (1.11) and evaluating the jump of the resulting relations we obtain

$$\begin{aligned} &k_{ijpq}[\dot{u}_{p,iq}] + \alpha a_{ij}[\ddot{\theta}_{,i}] + a_{ij}[\dot{\theta}_i] = \rho_0[\ddot{u}_j], \\ &a_{ij}[\ddot{u}_{i,j}] + k_{ij}[\dot{\theta}_{,ij}] + 2b_i[\ddot{\theta}_{,i}] - d[\ddot{\theta}] = h[\ddot{\theta}]. \end{aligned} \tag{2.17}$$

Multiplying (2.17) by (f_{qj}), summing on j, and using (2.2), (2.12), (2.14) and (2.16) we get

$$\Psi_{p,q} V_{,n_q} + A_{pq} \Psi_q = 0, \tag{2.18}$$

where

$$\begin{aligned} A_{ps} = V^2\{&k_{ijmq}\{f_{sm,q} n_i V^{-1} + (f_{sm} n_q V^{-1})_{,i}\} f_{pj} - a_{ij}\{\alpha(f_{s4,i} f_{pj} + \\ &+ f_{si,j} f_{p4}) - f_{s4} f_{pj} n_i V^{-1}\} + \alpha k_{ij}\{(f_{s4} n_i V^{-1})_{,j} + f_{s4,i} n_j V^{-1}\} f_{p4} - \\ &- 2\alpha b_i f_{s4,i} f_{p4} + \alpha d f_{s4} f_{p4}\}. \end{aligned}$$

Following the procedure given in [6] the Eqs. (2.18) may be written as an ordinary differential equation along the bicharacteristic

$$\frac{d\Psi_p}{dt} + A_{pq} \Psi_q = 0, \quad (p = 1, 2, \ldots, \beta). \tag{2.19}$$

Eqs. (2.19) are the governing differential equations of the amplitude of a isothermal acceleration wave propagating in the medium defined by the constructive relations (1.6).

The global behavior of a solution to the system (2.19) has been discussed by Nistor [6].

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UNDE DE ACCELERAŢIE ÎN MATERIALE TERMOELASTICE LINIARE

(Rezumat)

Se studiază propagarea undelor de acceleraţie în teoria liniară a termoelasticităţii stabilită de Green şi Lindsay [1]. Se arată că undele de acceleraţie ce se propagă într-o regiune cu starea uniformă sînt isotermice. Se stabileşte condiţia de propagare, apoi, folosind metoda bicaracteristicelor, se obţine ecuaţia de transport.

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VARIATIONAL THEOREMS IN THE THEORY OF ELASTIC MEDIA WITH MICROSTRUCTURE

BY

D. MANOLE

0. Introduction. Variational theorems in the theory of micropolar linear elasticity are given in [2], [3]. In the case of the linear theory of elastic media with microstructure a variational theorem of the Gurtin type is given in [1] and one of the Ignczak type in [4].

We shall further on present some variational theorems in the linear theory of elastic media with microstructure.

1. Basic Equations. Consider an elastic medium with microstructure occupying the Ω domain of the three-dimensional Euclidian space with closure $\bar{\Omega}$ and (regular) boundary Σ. The basic equations of linear elasticity theory with microstructure are [1]:

i) *equations of motion*

$$(\tau_{ji}+\sigma_{ji})_{,j}+\rho f_i=\rho\ddot{u}_i,$$
$$\mu_{kij,k}+\sigma_{ij}+\rho l_{ij}=I_{is}\ddot{\psi}_{sj}\,; \qquad \tau_{ij}=\tau_{ji}\,; \tag{1.1}$$

ii) *geometrical equations*

$$\varepsilon_{ij}=\tfrac{1}{2}(u_{j,i}+u_{i,j}), \qquad \gamma_{ij}=u_{j,i}-\psi_{ij}, \qquad \varkappa_{ijk}=\psi_{jk,i}\,; \tag{1.2}$$

iii) *constitutive equations*

$$\tau_{ij}=C_{ijkl}\varepsilon_{kl}+G_{klij}\gamma_{kl}+F_{pqrij}\varkappa_{pqr},$$
$$\sigma_{ij}=G_{ijkl}\varepsilon_{kl}+B_{klij}\gamma_{kl}+D_{ijpqr}\varkappa_{pqr}, \tag{1.3}$$
$$\mu_{ijk}=F_{ijkrs}\varepsilon_{rs}+D_{rsijk}\gamma_{rs}+A_{ijkpqr}\varkappa_{pqr}.$$

The new notations are : u_i—components of the displacement vector ; ψ_{ij}—components of micro-displacement tensor ; $\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}$—kinematic characteristics of the strain ; f_i—components of body force ; l_{ij}—components of body microforce ; τ_{ij}—components of the classical stress tensor ; σ_{ij}—components of the relative stress tensor ; μ_{ijk}—components of the couple-stress tensor ; $\rho(x)$—the mass density ; $I_{ij}(x)$, $C_{ijkl}(x)$, $G_{ijkl}(x)$, $B_{ijkl}(x)$, $D_{ijkl}(x)$, $F_{pqrij}(x)$, $A_{ijkpqr}(x)$—characteristic functions of the material ; the comma denote partial derivation with respect to the space variables x_i and a dot denotes partial derivation with respect to the time t.

The characteristic functions of the material obey the symmetry relations

$$I_{ij}=I_{ji},\quad C_{ijkl}=C_{klij}=C_{jikl},\quad B_{ijkl}=B_{klij},$$
$$A_{ijkpqr}=A_{pqrijk},\quad F_{ijkrs}=F_{ijksr},\quad G_{ijkl}=G_{ijlk}. \tag{1.4}$$

To the system of field equations we adjoin the initial conditions

$$u_i(x,0)=a_i,\quad \dot{u}_i(x,0)=b_i,\quad \psi_{ij}(x,0)=c_{ij},\quad \dot{\psi}_{ij}(x,0)=d_{ij} \tag{1.5}$$

and the boundary conditions

$$u_i=\tilde{u}_i \text{ on } \Sigma_1\times[0,t),\quad t_i=(\tau_{ji}+\sigma_{ji})n_j=\tilde{t}_i \text{ on } \Sigma_2\times[0,t),$$
$$\psi_{ij}=\tilde{\psi}_{ij} \text{ on } \Sigma_3\times[0,t),\quad T_{ij}=\mu_{kij}n_k=\tilde{T}_{ij} \text{ on } \Sigma_4\times[0,t), \tag{1.6}$$

where a_i, b_i, c_{ij}, d_{ij}, $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\psi}_{ij}$, $\tilde{T}_{ij}$ —are prescribed functions, n_i —components of the unit outward normal to Σ and $\Sigma_r (r=1, 2, 3, 4)$ are parts of Σ such that $\Sigma_1\cup\Sigma_2=\Sigma_3\cup\Sigma_4=\Sigma$, $\Sigma_1\cap\Sigma_2=\Sigma_3\cap\Sigma_4=0$.

2. Variational Theorems. We define as an admissible state $s=\{u_i, \psi_{ij}, \varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}, \tau_{ij}, \sigma_{ij}, \mu_{ijk}\}$ the ordered assembly of functions u_i, ψ_{ij}, ε_{ij}, γ_{ij}, $\varkappa_{ijk}$, τ_{ij}, σ_{ij}, $\mu_{ijk}\}$ defined on $\bar{\Omega}\times[0,t)$ with the properties

$$u_i,\ \psi_{ij}\in C^{1,2},\quad \varepsilon_{ij},\ \gamma_{ij},\ \varkappa_{ijk}\in C^{0,0},\quad \tau_{ij},\ \sigma_{ij},\ \mu_{ijk}\in C^{1,0}.$$

If we define the addition of the admissible states and the multiplication of an admissible state by a scalar as

$$s+s'=\{u_i+u_i',\ \psi_{ij}+\psi_{ij}',\ \varepsilon_{ij}+\varepsilon_{ij}',\ \gamma_{ij}+\gamma_{ij}',\ \varkappa_{ijk}+\varkappa_{ijk}',\ \tau_{ij}+\tau_{ij}',\ \sigma_{ij}+\sigma_{ij}',\ \mu_{ijk}+\mu_{ijk}'\},$$
$$\alpha s==\{\alpha u_i,\ \alpha\psi_{ij},\ \alpha\varepsilon_{ij},\ \alpha\gamma_{ij},\ \alpha\varkappa_{ijk},\ \alpha\tau_{ij},\ \alpha\sigma_{ij},\ \alpha\mu_{ijk}\},\quad \forall\alpha\sim\mathbf{R},$$

we find that the set of all admissible states is a linear space.

We call solution of the problem an admissible state satisfying Eqs. (1.1)—(1.3), initial conditions (1.5) and boundary conditions (1.6).

Theorem 2.1 [1]. *The functions u_i, ψ_{ij}, τ_{ij}, σ_{ij}, μ_{ijk} satisfy the equations* (1.1) *and the initial conditions* (1.5) *if and only if*

$$g*(\tau_{ji}+\sigma_{ji})_{,j}+F_i=\rho u_i,$$
$$g*(\mu_{kij,k}+\sigma_{ij})+L_{ij}=I_{is}\psi_{sj},\quad on\ \ \Omega\times[0,t), \tag{2.1}$$

where

$$g(t)=t,\qquad F_i=g*\rho f_i+\rho(tb_i+a_i),$$
$$L_{ij}=g*\rho l_{ij}+I_{is}(td_{sj}+c_{sj}),\qquad t>0. \tag{2.2}$$

Theorem 2.2. *An admissible state s can be considered solution of the mixed problem if and only if it satisfies equations* (2.1), (1.2), (1.3), (1.6).

If we introduce the notations

$$A_iu=\rho u_i-g*[C_{ijkl}u_{l,k}+B_{klij}(u_{i,j}-\psi_{ji})+G_{klij}(u_{l,k}-\psi_{kl})+$$
$$+G_{jikl}u_{l,k}+(F_{pqrji}+D_{jipqr})\psi_{qr,p}]_{,j},$$
$$A_{ln}u=I_{ls}\psi_{sn}-g*[(F_{klnrs}u_{s,r}+D_{rskln}(u_{s,r}-\psi_{rs})+A_{klnpqr}\psi_{qr,p})_{,k}+$$
$$+G_{lnks}u_{s,k}+B_{ksln}(u_{s,k}-\psi_{sk})+D_{lnpqr}\psi_{qr,p}],\quad (i,l,n=1,2,3), \tag{2.3}$$

from (2.1) we have

$$A_i u = F_i, \quad A_{ln} u = L_{ln}, \tag{2.4}$$

or

$$Au = f, \tag{2.5}$$

where

$$Au = (A_i u,\ A_{ln} u), \quad f = (F_i,\ L_{ln}).$$

For vectors $h = (h_1, h_2, \ldots, h_{12})$, $g = (g_1, g_2, \ldots, g_{12})$ we denote

$$h * g = \sum_{i=1}^{12} h_i * g_i. \tag{2.6}$$

If in reciprocity relation from [1] we introduce notations

$$u = (u_j^{(1)},\ \psi_{ik}^{(1)}), \quad v = (u_i^{(2)},\ \psi_{jk}^{(2)}),\ t(u) = (t_i^{(1)},\ T_{jk}^{(1)}),\ t(v) = (t_i^{(2)},\ T_{jk}^{(2)}); \tag{2.7}$$

on the basis of relations (2.5), (2.6), we get

$$\int_\Omega (v * Au - u * Av)\, d\Omega = \int_\Sigma g * [u * t(v) - v * t(u)]\, d\sigma. \tag{2.8}$$

Introducing the "convolution scalar product" of h and g functions, defined by [2]

$$(h \circledast g) = \int_\Omega h * g\, d\Omega, \tag{2.9}$$

relation (2.8) becomes

$$(Au \circledast v) - (u \circledast Av) = \int_\Sigma g * [u * t(v) - v * t(u)]\, d\sigma. \tag{2.10}$$

We consider the case when boundary conditions are homogeneous. From (2.10) it results that operator A is symmetric in convolution. Consequently, for the functional

$$F(u) = (Au \circledast u) - 2(u \circledast f), \tag{2.11}$$

we have

$$\delta F(u) = 0, \quad \text{on} \quad D_A \tag{2.12}$$

if and only if $u \in D_A$ satisfies equations (2.5), D_A being the definition domain of operator A.

It is observed that

$$\begin{aligned}(Au * u) = \int_\Omega g * [&C_{ijkl}\varepsilon_{ij} * \varepsilon_{kl} + B_{ijkl}\gamma_{ij} * \gamma_{kl} + A_{ijkpqr}\varkappa_{ijk} * \varkappa_{pqr} + \\ &+ 2G_{ijkl}\gamma_{ij} * \varepsilon_{kl} + 2F_{ijkrs}\varkappa_{ijk} * \varepsilon_{rs} + 2D_{ijpqr}\gamma_{ij} * \varkappa_{pqr}]\, d\Omega + \\ &+ \int_\Omega (\rho u_i * u_i + I_{js}\psi_{sk} * \psi_{jk})\, d\Omega.\end{aligned} \tag{2.13}$$

Taking into account (2.11) and (2.13) we get

$$\Gamma(u)=\int_{\Omega} g*[C_{ijkl}\varepsilon_{ij}*\varepsilon_{kl}+B_{ijkl}\gamma_{ij}*\gamma_{kl}+A_{ijkpqr}\varkappa_{ijk}*\varkappa_{pqr}+$$

$$+2G_{ijkl}\gamma_{ij}*\varepsilon_{kl}+2F_{ijkrs}\varkappa_{ijk}*\varepsilon_{rs}+2D_{ijpqr}\gamma_{ij}*\varkappa_{pqr}]\mathrm{d}\Omega+ \tag{2.14}$$

$$+\int_{\Omega}(\rho u_i*u_i+I_{js}\psi_{sk}*\psi_{jk})\mathrm{d}\Omega-2\int_{\Omega}(u_i+F_i+\psi_{jk}*L_{jk})\mathrm{d}\Omega,$$

where from it results

Theorem 2.3. *Let $\mathcal{M}\subset D_A$ be the vector set $u=(u_i, \psi_{jk})$ satisfying conditions (1.6) in a homogeneous form. Then*

$$\delta\Gamma(u)=0, \text{ on } \mathcal{M} \tag{2.15}$$

if and only if $u\in\mathcal{M}$ is a solution of a mixed problem.

We consider the case of inhomogeneous boundary conditions. Let $K\subset D_A$ be the vector set $u=(u_i, \psi_{jk})$ satisfying conditions

$$u_i=\tilde{u}_i \text{ on } \Sigma_1\times[0, t), \quad \psi_{ik}=\tilde{\psi}_{jk} \text{ on } \Sigma_3\times[0, t). \tag{2.16}$$

In this case we get the following functional

$$\Lambda(u)=\int_{\Omega} g*[C_{ijkl}\varepsilon_{ij}*\varepsilon_{kl}+B_{ijkl}\gamma_{ij}*\gamma_{kl}+A_{ijkpqr}\varkappa_{ijk}*\varkappa_{pqr}+$$

$$+2G_{ijkl}\gamma_{ij}*\varepsilon_{kl}+2F_{ijkrs}\varkappa_{ijk}*\varepsilon_{rs}+2D_{ijpqr}\gamma_{ij}*\varkappa_{pqr}]\mathrm{d}\Omega+ \tag{2.17}$$

$$+\int_{\Omega}\rho u_i*u_i+I_{js}\psi_{sk}*\psi_{jk})\mathrm{d}\Omega-2\int_{\Omega}(u_i*F_i+\psi_{jk}*L_{jk})\mathrm{d}\Omega-$$

$$-2\int_{\Sigma_2} g*\tilde{t}_i*u_i\mathrm{d}\sigma-2\int_{\Sigma_4} g*\tilde{T}_{jk}*\psi_{jk}\mathrm{d}\sigma.$$

and have

Theorem 2.4. *The vector $u\in K$ is a solution of a mixed problem if and only if*

$$\delta\Lambda(u)=0, \text{ on } K. \tag{2.18}$$

Consider $s=\{u_i, \psi_{ij}, \varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}, \tau_{ij}, \sigma_{ij}, \mu_{ijk}\}$ an admissible state and the functional

$$J_t(s)=\tfrac{1}{2}[\rho\dot{u}_i, \dot{u}_i]+\tfrac{1}{2}[I_{ij}\dot{\psi}_{is}, \psi_{sj}]+\tfrac{1}{2}[\tau_{ij}, u_{j,i}+u_{i,j}-2\varepsilon_{ij}]+$$

$$+[\sigma_{ij}, u_{j,i}-\psi_{ij}-\gamma_{ij}]+[\mu_{ijk}, \psi_{jk,i}-\varkappa_{ijk}]-[\rho f_i, u_i]-[\rho l_{ij}, \psi_{ij}]+ \tag{2.19}$$

$$+\tfrac{1}{2}[C_{ijkl}\varepsilon_{kl}+G_{klij}\gamma_{kl}+F_{pqrij}\varkappa_{pqr}, \varepsilon_{ij}]+\tfrac{1}{2}[G_{ijkl}\varepsilon_{kl}+B_{klij}+D_{ijpqr}\varkappa_{pqr}, \gamma_{ij}]+$$

$$+\tfrac{1}{2}[F_{ijkrs}\varepsilon_{rs}+D_{rsijk}\gamma_{rs}+A_{ijkpqr}\varkappa_{pqr}, \varkappa_{ijk}]-[t_i, u_i-\tilde{u}_i]_{\Sigma_1}-[\tilde{t}_i, u_i]_{\Sigma_2}-$$

$$-[T_{ij}, \psi_{ij}-\tilde{\psi}_{ij}]_{\Sigma_3}-[\tilde{T}_{ij}, \psi_{ij}]_{\Sigma_4}+[\rho(\dot{u}_i-b_i), u_i]_0-$$

$$-[\rho(u_i-2a_i), \dot{u}_i]_0+[I_{ij}(\dot{\psi}_{ij}-d_{is}, \psi_{sj}]_0-[I_{ij}(\psi_{is}-2C_{is}), \dot{\psi}_{sj}]_0$$

defined on the set of the admissible states, where

$$[h, g]=\int_\Omega\int_0^t h(x, \tau)g(x, t-\tau)\mathrm{d}\tau\mathrm{d}\Omega,$$

$$[h, g]_0=\int_\Omega \{h(x, \tau)g(x, t-\tau)\}_{\tau=0}^t\mathrm{d}\Omega, \tag{2.20}$$

$$[h, g]_\Sigma=\int_\Sigma\int_0^t h(x, \tau)g(x, t-\tau)\mathrm{d}\tau\mathrm{d}\sigma.$$

Theorem 2.5. *Let* $s=\{u_i, \psi_{ij}, \varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}, \tau_{ij}, \sigma_{ij}, \mu_{ijk}\}\in K$ *and for* $t\in[0, \infty)$ *define the functional* $J_t(s)$ *on* K *through* (2.19). *Then*

$$\delta J_t(s)=0 \tag{2.21}$$

if and only if s *is a solution of the mixed problem.*

Proof. Consider $s=\{u_i, \psi_{ij}, \varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}, \tau_{ij}, \sigma_{ij}, \mu_{ijk}\}$ and $s'=\{u'_i, \psi'_{ij}, \varepsilon'_{ij}, \gamma'_{ij}, \varkappa'_{ijk}, \tau'_{ij}, \sigma'_{ij}, \mu'_{ijk}\}$ two admissible states. It is obvious $s+\alpha s'\in K$ for every real $\alpha\in\mathbb{R}$. If we can form the expression

$$\delta_{s'}J_t(s)=\frac{\mathrm{d}}{\mathrm{d}\alpha}J_t(s+\alpha s')\Big|_{\alpha=0} \tag{2.22}$$

we can deduce that

$$\begin{aligned}
\delta_{s'}J_t(s)=&[\rho\dot{u}_i, \dot{u}'_i]+]I_{ij}\dot{\psi}_{is}, \dot{\psi}'_{sj}]+\tfrac{1}{2}[\tau'_{ij}, u_{j,i}+u_{i,j}-2\varepsilon_{ij}]+\\
&+\tfrac{1}{2}[\tau_{ij}, u'_{j,i}+u'_{i,j}-2\varepsilon'_{ij}]+[\sigma'_{ij}, u_{j,i}-\psi_{ij}-\gamma_{ij}]+[\sigma_{ij}, u'_{j,i}-\psi'_{ij}-\gamma'_{ij}]+\\
&+[\mu'_{ijk}, \psi_{jk,i}-\varkappa_{ijk}]+[\mu_{ijk}, \psi'_{jk,i}-\varkappa'_{ijk}]-[\rho f_i, u'_i]-[\rho l_{ij}, \psi'_{ij}]+\\
&+[C_{ijkl}\varepsilon_{kj}+G_{klij}\gamma_{kl}+F_{pqrij}\varkappa_{pqr}, \varepsilon'_{ij}]+[G_{ijkl}\varepsilon_{kl}+B_{klij}\gamma_{kl}+\\
&+D_{ijpqr}\varkappa_{pqr}, \gamma'_{ij}]+[F_{ijkrs}\varepsilon_{rs}+D_{rsijk}\gamma_{rs}+A_{ijkpqr}, \varkappa'_{ijk}]-[l'_i, u_i-\tilde{u}_i]_{\Sigma_1}-\\
&-[t_i, u'_i]_{\Sigma_1}-[\tilde{t}_i, u'_i]_{\Sigma_2}-[T'_{ij}, \psi_{ij}-\tilde{\psi}_{ij}]_{\Sigma_3}-[\tilde{T}_{ij}, \psi'_{ij}]_{\Sigma_4}+\\
&+[\rho(\dot{u}_i-b_i), u'_i]_0+[\rho\dot{u}'_i, u_i]_0-[\rho(u_i-2a_i), \dot{u}'_i]_0-[\rho u'_i, \dot{u}_i]_0+\\
&+[I_{ij}(\dot{\psi}_{is}-d_{is}), \psi'_{sj}]_0+[I_{ij}\dot{\psi}'_{is}, \psi_{sj}]_0-[I_{ij}(\psi_{is}-2C_{is}), \dot{\psi}'_{sj}]_0-[I_{ij}\psi'_{is}, \dot{\psi}_{sj}]_0.
\end{aligned} \tag{2.23}$$

Integrating by parts and taking into account the divergence theorem we get

$$\begin{aligned}
\delta_{s'}J_t(s)=&\tfrac{1}{2}[\tau'_{ij}, u_{j,i}+u_{i,j}-2\varepsilon_{ij}]+[\sigma'_{ij}, u_{j,i}-\psi_{ij}-\gamma_{ij}]+[\mu'_{ijk}, \psi_{jk,i}-\varkappa_{ijk}]+\\
&+[\rho\ddot{u}_i-(\tau_{ji}+\sigma_{ji})_{,j}-\rho f_i, u'_i]+[I_{ij}\ddot{\psi}_{is}-\mu_{kis,k}-\sigma_{is}-\rho l_{is}, \psi'_{is}]+
\end{aligned} \tag{2.24}$$

$$+[C_{ijkl}\varepsilon_{kl}+G_{klij}\gamma_{kl}+F_{pqrij}\varkappa_{pqr}-\tau_{ij},\ \varepsilon'_{ij}]+$$
$$+[G_{ijkl}\varepsilon_{kl}+B_{klij}\gamma_{kl}+D_{ijpqr}\varkappa_{pqr}-\sigma_{ij},\ \gamma'_{ij}]+$$
$$+[F_{ijkrs}\varepsilon_{rs}+D_{rsijk}\gamma_{rs}+A_{ijkpqr}\varkappa_{pqr}-\mu_{ijk},\ \varkappa'_{ijk}]-[t'_i,\ u_i-\tilde{u}_i]_{\Sigma_1}+$$
$$+[t_i-\tilde{t}_i,\ u'_i]_{\Sigma_2}-[T'_{ij},\ \psi_{ij}-\tilde{\psi}_{ij}]_{\Sigma_3}+[T_{ij}-\tilde{T}_{ij},\ \varphi'_{ij}]_{\Sigma_4}+$$
$$+[\rho(\dot{u}_i-b_i),u'_i]_0-2[\rho(u_i-a_i),\ \dot{u}'_i]_0+[I_{ij}(\dot{\psi}_{is}-d_{is}),\ \psi'_{sj}]_0-$$
$$-2[I_{ij}(\psi_{is}-C_{is}),\ \dot{\psi}'_{sj}]_0.$$

If s is a solution of the mixed problem, then from (2.24) it results

$$\delta_{s'}J_t(s)=0,\quad \forall s'\in K \tag{2.25}$$

and thus (2.21) takes place ($\delta'_s J_t(s)$) being linear in s').

Inversely, we suppose that relation (2.21) takes place, that implying (2.25). It is obvious that $\mathring{s}=\{0,0,0,0,0,0,0,0\}$ and thus we can select $s'=\{u'_i,0,0,0,0,0,0,0\}$ so that u'_i, together with all of its space derivatives, vanish on $\Sigma\times[0,t)$. Thus (2.24) and (2.25) with the selection made implies

$$[\rho\ddot{u}_i-(\tau_{ji}+\sigma_{ji})_{,j}-\rho f_i,\ u'_i]=0,\quad t\geqslant 0 \tag{2.26}$$

and on the basis of the fundamental lemma of the calculus of variations [5] it results

$$(\tau_{ji}+\sigma_{ji})_{,j}+\rho f_i-\rho\ddot{u}_i=0. \tag{2.27}$$

Consider the same choice of s' but require now that u'_i vanish on $\Sigma_1\times[0,t)$. Then (2.24), (2.25), (2.27) we have

$$[t_i-\tilde{t}_i,\ u'_i]_{\Sigma_2}=0, \tag{2.28}$$

where from, generalizing the fundamental lemma, it results

$$t_i=\tilde{t}_i\quad\text{on}\quad \Sigma_2\times[0,t). \tag{2.29}$$

Now let $s'=\{0,\psi'_{ij},\ 0,0,0,0,0,0\}$ and suppose ψ'_{ij} and all of its space derivatives, vanish on $\Sigma\times[0,t)$.

Then (2.24), (2.25) and the fundamental lemma, we get

$$\mu_{kij,k}+\sigma_{ij}+\rho l_{ij}=I_{is}\ddot{\psi}_{sj}. \tag{2.30}$$

Next let $s'=\{0,\psi'_{ij},\ 0,0,0,0,0,0\}$ but this time requires that vanish on $\Sigma_3\times[0,t)$; we get

$$T_{ij}=\tilde{T}_{ij}\quad\text{on}\quad \Sigma_4\times[0,t). \tag{2.31}$$

Continuing in this way there results the proof of the theorem.

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TEOREME VARIAŢIONALE ÎN TEORIA ELASTICITĂŢII LINIARE CU MICROSTRUCTURĂ

(Rezumat)

Se prezintă cîteva teoreme variaţionale în teoria elasticităţii liniare cu microstructură.

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DES NOUVEAUX ASPECTS SUR LA VITESSE DE PROPAGATION DES INTERACTIONS

PAR

GH. ZET, MARCEL AGÔP et I. VULPOI-NAGHEL

0. Introduction. On fait une analyse du processus de mesure dans T.R.R. Le résultat de celle-ci montre que le champ électromagnétique n'est pas plus adéquat aux mesurages, mais le champ „gauge" qui serve comme médiateur pour les interactions.

1. Les incertitudes en distances et le signal de synchronisation. Dans T.R.R. la distance est déterminée à l'aide d'un signal lumineux envoyé de l'origine du repère et reflété (renvoyé) instantanément dans le point auquel on veut établir la distance en rapport avec la même origine. Conseignant le moment de l'envoi et de la réception du signal après qu'il a été reflété on obtient l'interval du temps dans lequel la lumière parcoure deux fois cette distance. En sachant la vitesse de la lumière on peut, donc, calculer cette distance [1].

Mais, ça constitue une abstraction théorique si on soumete l'expérimentation aux raisonnement phénoménologique. Premièrement, on doit s'assurer de la constance de la vitesse de la lumière sur tout le parcours, dans l'interval du temps mesuré. Si, par exemple, entre l'origine et le point considéré l'espace n'est pas homogène dans l'interval du temps pendant lequel on mesure, alors la condition de la constance de la vitesse de la lumière n'est pas satisfaite. La déterminaison de la distance se fait avec la valeur locale de la vitesse de la lumière, ce qui conduit à l'incertitude en distance. Admettons pourtant qu'on est prudent et qu'on a analysé d'une manière qulconque un voisinage de l'origine du repère en s'assurant que pour un interval de temps suffisamment grand rien n'influence la propagation de la lumière. Dans ce cas, la distance aussi ne pêut pas être donnée toujours à cause que la réflexion de la lumière ne pêut pas être à notre disposition en chaque point de ce voisinage. Plus général, la réflexion pêut être considérée un phénomène impossible dans le voisinage choisie. Par exemple, dans l'électrodinamique cuantique un phénomène localisable pourait être la réalisation des paires à laquelle on poura rapporter un schéma analogue à ce relativiste. Donc, on rencontre un problème de plus : manque le fait que le signal de réponse pêut être d'autre nature que celle du signal envoyé, il est possible qu'il n'existe pas ou qu'il est transmis en autre direction. Tout de même, c'est clair que le schéma relativiste doit être en désaccord avec l'électrodinamique quantique. Ce désaccord est responsable de la nécésité de la renormalisation [2].

Celles antérieures sugèrent le fait que si dans un repère cartésien on veut établir une distance la procédure et la théorie doivent être suffisant d'élas-

tiques pour accepter les auxiliaires ci-dessous : 1° un signal envoyé de l'origine du repère ; 2° un signal du réponse du point avisé auquel ou veut calculer la distance ; 3° un horloge pour marquer le moment d'envoi et de la réception des signals.

Ces deux signals peuvent être différentiés par leur nature, ou un d'eux pêut manquer au mode réel. On ne pêut pas supposer, par exemple, que quelqu'un certain de sur la Terre a pu provoqué la radiation emise à un certain moment par un pulsar ! Pourtant elle pêut être analysée implicitement, elle engendre certaines conclusions et elle nous donne une estimation de la distance. De plus, quand tous les deux signals existent au point visé, entre la réception d'un et le renvoi de l'autre, il pêut exister une entière série de phénomènes, ce que fait que le signal du réponse est renvoié en retard, retardement duquel on ne pêut pas, en général, tenir compte dans l'origine du repère. Ce qui détermine, évidemment, une autre indétermination en distance.

Pour l'instant, essayons de préciser la position d'un champ nonperturbé [1] dans le contexte présente. Reprennons donc la discussion de l'expérimentation du messurage d'une distance, de T.T.R. à la seule différence qu'on ne parlera pas de rayon de lumière, mais de signal envoyé et de signal du réponse, comme on a convenu. On admet qu'on connaît les vitesses de propagation de ces deux signals et on les notte avec c_1 et c_2. En ce cas on peut donner une estimation de la distance seulement en fonction des valeurs connues c_1 et c_2 et l'interval du temps Δt marqué par l'horloge affecté au repère, entre l'envoi et la réponse. Hypothétiquement, on écrit

$$\Delta t = (\Delta t)_1 + (\Delta t)_2, \tag{1}$$

ou $(\Delta t)_1$ et $(\Delta t)_2$ sont respectivement, les intervalles de temps nécessaires aux deux signals pour couvrir la distance considérée. Si on note avec d cette distance, on aura, aussi hypothétiquement

$$(\Delta t)_1 = \frac{d}{c_1}; \quad (\Delta t)_2 = \frac{d}{c_2}. \tag{2}$$

Après un calcule simple, on obtient la relation

$$d = \frac{c_1 c_2}{c_1 + c_2} \Delta t. \tag{3}$$

Évidemment, la relation (3) offre seulement une estimation soumise aux incertitudes mentionnées : la relation (1) n'est pas valable exactement ; elle doit être amendée avec l'interval du temps passé au point de l'espace considéré, entre la reception du signal transmis de l'origine et l'envoi de l'autre à l'origine, et les relations (2) ne sont pas valables que localement. Évidemment que chaque cas particulier de la formule (3) sera soumis aux même incertitudes. De cette formule le cas relativiste s'obtient si $c_1 = c_2 = c$ et

$$d = \frac{c\Delta t}{2}. \tag{4}$$

Mais, le plus important c'est d'analyser le cas où l'une des valeurs c_1 et c_2 et beaucoup plus grande que l'autre (pratiquement, l'un des signals se propaguant „instantanément"). Donc, la formule (3) donne

(5) $$d = c\Delta t,$$

où c représente cette fois-ci la moindre valeur de c_1 et c_2. Sous cette forme on accèpte en général la distance dans T.R.R.

Mais, en ce cas, l'accent tombe sur la connaissance de l'intervalle de temps et de la vitesse du signal considéré. Donc le problème peut avoir un nouveau aspect calitatif : celui de la possibilité de synchroniser les horloges trouvés dans des différents points de l'espace. Car on peut offrir un test objectif de cette synchronisation, si l'on connaît la nature physique des deux signals.

Vraiment, admettons que les signals dans l'origine du repère et dans le point dont on veut calculer la distance sont admis par les horloges dans les points donnés directement (saus aucune intervention subjective).

Alors, si au signal envoyé connu, l'observateur de l'origine reçoit la réponse attendue, de manière que (5) soit le plus exactement possible (mathématiquement parlant) il peut déclarer que les deux horloges sont synchronisés. Il est évident que cette formule a des limites de valabilité, imposées par les incertitudes physiques considérées antérieurement. En plus, on doit mentionner que le domaine de valabilité de d pour lequel la synchronisation conçue existe, est limité physiquement, pour des valeurs qui dépassent une certaine limite de distance, le signal qu'on présupose instantané pourrait ne plus satisfaire cette condition avec l'aproximation mathématique prévue.

Il est évident qú un tel signal caractérise un certain voisinage de l'origine, le voisinage pour qui les signals précisés dans le repère donné sont ceux qui déterminent la distance par l'existence indépendante de l'isotropie, de l'homogeneité spatielle etc., c'est à dire les qualités du champ non-perturbé mentionné.

Donc, le champ nonperturbé sur un voisinage donné représente le signal de synchronisation. Mais, sur un voisinage, le signal lumineux ne peut pas avoir toujours ces qualités.

2. Les champs „gauge" comme signal de synchronisation. Le plus banal cas où la vitesse limite v est $\neq c$ est celui des particles dont la masse de repos $m_0 \neq 0$. Pour ce champ, l'équation de propagation de l'interaction est donnée par l'équation de d'Alembert généralisée

(6) $$(\Box - k^2)\psi = 0\,; \qquad k \sim m_0,$$

où

(7) $$\Box = \Delta - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\,.$$

Si l'on admet que

(8) $$\psi = \psi_0(\mathbf{r})e^{i\omega t},$$

alors

(9) $$k^2\psi = -\frac{k^2}{\omega^2}\,\frac{\partial^2\psi}{\partial t^2}\,.$$

Donc (6) prends la forme

(10) $$\Delta\psi - \frac{1}{v^2}\,\frac{\partial^2\psi}{\partial t^2} = 0\,,$$

ou v a donc l'expression

$$v = \frac{c\omega}{\sqrt{\omega^2 + k^2}}. \tag{11}$$

L'apparition de la masse conduit ainsi a l'existence d'une vitesse limite $\neq c$. Dans ce cas-là, le champ mésonique a les qualités du champ non-perturbé.

Les théories „gauge" montrent que le champ électromagnétique est un cas particulier de champ „gauge" (A^a_μ; $\mu = \overline{1, 4}$; $a = 1, 2, \ldots$) [3]. Dans ces conditions on peut adopter le postulat : *la vitesse de propagation des interactions est celle du „champ gauge" qui moyenne ces interactions.*

Donc, dans le cas particulier de l'interaction électron-électron, le champ „gauge" est celui électromagnétique (A_μ) et la vitesse de propagation est c.

Dans la cas des interactions neutron-proton, le champ „gauge" est celui mésonique, et la vitesse et donnée par la formule (11).

Parce que dans la nature les interactions sont formées d'un mélange de champs (on ne peut pas trouver les champs dans un état pur), la vitesse limite sera celle du signal nonperturbé.

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NOI ASPECTE PRIVIND VITEZA DE PROPAGARE A INTERACŢIUNILOR

(Rezumat)

Pornind de la analiza procesului de măsură în T.R.R. se arată că cel mai adecvat semnal de sincronizare pe orice vecinătate a unui reper este cîmpul „gauge" ce mediază interacţiunile, întrucît el păstrează calităţile de omogenitate, izotropie etc. În acest caz cîmpul electromagnetic sau mezonic se dovedesc doar cazuri particulare ale unei asemenea tratări.

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A THEORY OF GRAVITATION BASED ON THE CONCEPT OF PHYSICAL FRAME

BY

MARCEL AGOP and GH. ZET

0. Introduction. One constructs mathematical orthogonal frames as solutions of a binary cubic. The apolarity condition selects the physical frame from the mathematical ones. As a consequence, the physical frame is defined by means of the spin parameters as three direction cosines (SU_2 algebra being also introduced).

The variational principle is applied under the hypothesis that there exists a direction parallelism of the frames. The spacelike and timelike invariants are both determined and some static metric are then constructed independently of Einstein's equations. The last are used only to give the structure of the energy-momentum tensor. The validity of the model is verified and the Lorentz transformations are implicitly obtained.

In our model, the gravitation arises in a natural way as an intrinsic property of the particle spin.

1. Mathematical and Physical Frames. The concept of orthogonal frame in an arbitrary point (chosen as origin) is so important that it is hard to think that it could be possible to replace it with anything else in physics. The important problem when we choose a frame is to establish space directions and to realise their orthogonality in the origin point. Then the way in which we associate a frame to a given point remains completely arbitrary, excepting the particular aspects of the physical problem which is considered. In other words, the association of an orthogonal frame to a given point of the space is a subjective thing.

Every orthogonal frame can be described from mathematical point of view by an orthogonal coordinate system. But, in this case it appears the following question. The cartesian coordinates are considered as observable variables in physics, i.e. they can be measured by an appropriate physical process. Now, excepting some extremely seldom cases, there is not possible to measure directly the cartesian coordinates. Instead, they can only be calculated analytically. For example, the position of a star is given in astronomy by two angles and the distance star—observer (not by its cartesian coordinates). In general, when we study experimentally the motion of a particle in a given field its position is described by a direction and by the distance particle—observer. When we exchange the cartesian coordinates to the spherical or cylindric coordinates, this not only simplifies the calculations but gives different

physical quantities as functions of observable variables. Examples are the Lorentz transformations in spherical coordinates [8] and the eigenvalues of some quantum operators in the case of a field with spherical symmetry [6].

On the other hand, in quantum mechanics there exists a transcendence of directions induced by Heisenberg relations. Indeed, the experimental apparatus must satisfies the classical laws of physics [10], even in the case of the quantum measurements. The difference between the classical and quantum laws, with respect to measure scale, arises for length, time, mass etc., but not for directions in space. A fixed direction at macroscopic level has physical significance in microphysics too. Therefore, looking for a connection between the classical and quantum theory of gravitation we must use frames defined only by directions.

The physical frame cannot be rotated arbitrary without an exchange of the phenomena described with respect to its directions. Therefore, such a frame must be chosen just as it appears in a given experiment, without the possibility of arbitrary rotations of its directions.

We will suppose then that a mathematical frame is given by thre orthogonal axis with their intersection in the origin point. The invariance under the rotations of the mathematical frames will select the physical frame.

2. Construction of Mathematical Frames. Differential Properties. Let us consider a 3-dimensional projective space with the homogeneous coordinates $x_i=(x_0, x_1, x_2, x_3)$, where a binary cubic is defined by the equation

$$x_0\lambda^3-3x_1\lambda^2+3x_2\lambda-x_3\equiv 0. \tag{1}$$

If we suppose that an arbitrary frame can be given as an intersection of three generic planes from the family

$$X_\alpha+\lambda Y_\alpha=0, \quad \alpha=1, 2, 3, \tag{2}$$

where X_α, Y_α are linear equations of the form

$$X_\alpha=a_{\alpha i}x_i\equiv 0\,; \qquad Y_\alpha=b_{\alpha i}x_i\equiv 0, \qquad i=\overline{0, 3}, \tag{3}$$

and λ is a parameter characterizing different planes, then one can prove that the coefficients of the cubic (1) are just the coordinates of the arbitrary frame and the solutions of the cubic are the coordinates of an orthogonal frame [1].

Therefore, we will consider an orthogonal frame characterized by the binary cubic (1) with real and distinct roots. The cubic can be written in the canonical form [1]

$$(\lambda-h)^3+k^3(\lambda-\bar{h})^3\equiv 0, \tag{4}$$

where h, $\bar{h}$ are the roots of its Hessian and k is a unimodular coefficient. The roots of the Eq. (4) are [1]

$$\lambda_i=\frac{h+\omega_i\bar{h}k}{1+\omega_i k}, \tag{5}$$

where ω_i are the cubic roots of the unity.

The most general transformation of a binary cubic is the homography of its variables

$$\lambda' = \frac{a\lambda + b}{c\lambda + d}; \quad a, b, c, d \in \mathbb{R}. \tag{6}$$

Under a homography transformation, the complex quantities (h, k) transform as [1]

$$h' = \frac{ah + b}{ch + d}; \quad k' = \frac{\bar{c}h + d}{ch + d} k. \tag{7}$$

These transformations define a transitive simple group of the cubic's parameters. The invariant 1-forms (Pfaff) associated with this group are [1]

$$\omega_0 = \mathrm{i}\left(\frac{\mathrm{d}k}{k} - \frac{\mathrm{d}h + \mathrm{d}\bar{h}}{h - \bar{h}}\right)^2; \quad \omega_1 = \bar{\omega}_2 = \frac{\mathrm{d}h}{k(h - \bar{h})}; \quad \mathrm{i} = \sqrt{-1}. \tag{8}$$

We give a physical significance of these 1-forms as follows. It is known that every binary cubic has its coefficients defined up to an arbitrary factor and its roots are invariant under such a normalization. Then, because a binary cubic is determined in a unique way by its coefficients and only up to a factor by its roots, it is possible to organize the set of the binary cubics as a 3-dimensional projective space. The coordinates of this space are just the coefficients of the binary cubic (1). The above group of transformations generates in this projective space linear transformations with coefficients depending on cubic's parameters. Therefore, the 1-forms (8) define an infinite set of metrics

$$\mathrm{d}s^2 = g_{ij}\omega_i\omega_j, \quad i, j = 1, 2, \tag{9}$$

that are invariant under the transformations (7). In the set of these metrics there is one of fundamental interest, namely that of quadratic form

$$\mathrm{d}s^2 = \omega_0^2 - 4\omega_1\omega_2. \tag{10}$$

This is an indefinite metric and its Riemannian curvature is constant.

If we take

$$h = u + \mathrm{i}v; \quad k = \mathrm{e}^{\mathrm{i}\psi}, \tag{11}$$

then the 1-forms (8) become

$$\omega_0 = -\mathrm{d}\psi + \frac{\mathrm{d}u}{v}; \quad \omega_1 = \frac{\mathrm{d}v - \mathrm{i}\,\mathrm{d}u}{2v}\mathrm{e}^{\mathrm{i}\psi}; \quad \omega_3 = -\frac{\mathrm{d}v + \mathrm{i}\,\mathrm{d}u}{2v}\mathrm{e}^{\mathrm{i}\psi} \tag{12}$$

and the metric (10) can be written as follows:

$$\mathrm{d}s^2 = -\left(\mathrm{d}\psi + \frac{\mathrm{d}u}{v}\right)^2 + \frac{\mathrm{d}u^2 + \mathrm{d}v^2}{v^2}. \tag{13}$$

3. Construction of Physical Frames. The Apolar Transport. The same group (7) can be also defined by means of the invariant 1-forms [6]

$$\begin{aligned} \mathrm{d}s^1 &= \frac{1}{v}(\cos\psi\,\mathrm{d}u + \sin\psi\,\mathrm{d}v); \quad \mathrm{d}s^2 = \frac{1}{v}(\sin\psi\,\mathrm{d}u - \cos\psi\,\mathrm{d}v); \\ \mathrm{d}s^3 &= \mathrm{d}\psi + \frac{\mathrm{d}u}{v}, \end{aligned} \tag{14}$$

which result from (12) after some simple algebraic transformations. If the 1-form ds^2 is a total differential, i.e.

$$d\psi = -\frac{du}{v} \tag{15}$$

then, the following differential operators can be constructed with [6]:

$$\hat{s}_1 = \left(\cos\psi\, v\frac{\partial}{\partial v} - \sin\psi\frac{\partial}{\partial\psi}\right); \quad \hat{s}_2 = \left(\sin\psi\, v\frac{\partial}{\partial v} + \cos\psi\frac{\partial}{\partial\psi}\right); \quad \hat{s}_3 = \mathrm{i}\frac{\partial}{\partial\psi}. \tag{16}$$

These operators are identical, up to a factor $\hbar$, with the angular momentum operators in the representations

$$x = v\sin\psi; \quad y = -v\cos\psi; \quad z = \mathrm{i}v. \tag{17}$$

Y a m a m o t o [9] have shown that the operators (17) are just the spin operators satisfying the same commutation relations as Pauli matrices σ_i ($i=1$, 2, 3). They can be interpreted as angular momentum operators in the complex space of null radius

$$x^2 + y^2 + z^2 = 0. \tag{18}$$

Now, the corresponding variables (v, ψ) are not concrete angular variables but just only internal freedom degrees.

Therefore, the condition (15) selects the physical frame from the mathematical ones, and the variables x, y, z when viewed as parameters of the spin space [9] define the direction cosines of the physical frame.

Then, an important problem is to show what the condition (15) becomes when the formalism of the frame (1) is used. We have the

T h e o r e m. *The apolar transport of cubic 1-forms is completely covered by the Levi-Civita transport in the Beltrami's metric of the Lobacevsky plane.*

To this end, we consider the binary cubic (1) and

$$x_0'\lambda^3 - 3x_1'\lambda^2 + 3x_2'\lambda - x_3' = 0. \tag{19}$$

The apolar of two cubic forms is an invariant of the form [1]

$$P \equiv x_3x_0' - 3x_2x_1' + 3x_1x_2' - x_0x_3' \tag{20}$$

and if P vanishes then the two cubics are in involution. In addition, if the cubics are infinitesimal separated, we can write

$$x_3\mathrm{d}x_0' - 3x_2\mathrm{d}x_1' + 3x_1\mathrm{d}x_2' - x_0\mathrm{d}x_3' = 0. \tag{21}$$

But

$$x_0 = 1 + k^3; \quad x_1 = -h + k^3\bar{h}; \quad x_2 = h^2 + k^3\bar{h}^2; \quad x_3 = h^3 - k^3\bar{h}^3$$

and then the apolarity condition is

$$-3k^3(h-\bar{h})^3\left\{\frac{\mathrm{d}k}{k} - \frac{\mathrm{d}h + \mathrm{d}\bar{h}}{h-\bar{h}}\right\} \equiv 0. \tag{22}$$

This means that $\omega_0 = 0$ and therefore we have obtained again (15).

Now, because in this case the metric (13) is of Beltrami form

$$ds^2 = \frac{du^2 + dv^2}{v^2}, \tag{23}$$

then we can show that the condition (15) gives the angle between two vectors obtained one another by a parallelism in the Lobacevsky plane [1]. As a consequence, we have

L e m m a. *The apolarity condition* (15) *transforms the group* (7) *into the rotation group* $O(3)$.

This shows how can we select the physical frame from the mathematical ones.

4. The Variational Principle. The General Solution of the Proposed Model. The metric (23) can be interpreted as a Riemannian metric and the corresponding geometry describes then some physical properties. Nevertheless, we will use another formalism, given in [7]; namely, every stationary gravitational field is characterized by a Lagrangian that is a function of two field variables (u, v) satisfying the variational principle

$$\delta \int \frac{\nabla h \nabla \bar{h}}{(h - \bar{h})^2} dv \equiv 0. \tag{24}$$

This form results from

$$\delta \int \left[4 \frac{\nabla h \nabla \bar{h}}{(h - \bar{h})^2} - \left(\frac{\nabla k}{k} - \frac{\nabla h + \nabla \bar{h}}{h - \bar{h}} \right)^2 \right] dv = 0, \tag{25}$$

if the condition (15) is used. In fact, a new symmetry is induced by the group (7) itself. This group acts on the hyperboloid of one sheet

$$\chi = x_0 x_3 - x_1 x_2 \equiv 0, \tag{26}$$

which is considered according to χ as an absolute of the binary cubic (1). Then, the Ernst principle [3] coincides with (24).

The variational principle seems to be of fundamental value because it gives directly the field invariants as functions of coordinates. Then, because (24) does not use the gravitational equations, we can construct the metric independently of Einstein's field equations. Namely, every 4-dimensional Riemannian metric

$$ds^2 = g_{ik} dx^i dx^k, \quad i, k = 1, 2, 3, 4 \tag{27}$$

can be written in the form

$$ds^2 = g_{44} \left(dx^4 + \frac{g_{\gamma\alpha} dx^\alpha}{g_{\gamma\gamma}} \right)^2 - \left(-g_{\alpha\beta} + \frac{g_{\gamma\alpha} g_{\gamma\beta}}{g_{\gamma\gamma}} \right) dx^\alpha dx^\beta. \tag{28}$$

This induces the spacelike quaadratic form

$$g'_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{\gamma\alpha} g_{\gamma\beta}}{g_{\gamma\gamma}} \tag{29}$$

and the linear form

$$g_\alpha = -\frac{g_{\gamma\alpha}}{g_{\gamma\gamma}}. \tag{30}$$

The expression (29) can be reduced to the canonical form by an appropriate transformation of coordinates, i.e.

$$g'_{\alpha\beta} dx^\alpha dx^\beta = \lambda_1 (dx^1)^2 + \lambda_2 (dx^2)^2 + \lambda_3 (dx^3)^2, \tag{31}$$

where λ_α are solutions of the binary cubic

$$| g'_{\alpha\beta} - \lambda \delta_{\alpha\beta} | = 0. \tag{32}$$

In the same time, the components g_α transform to the form (17). Therefore, using the variational principle we can construct both spacelike and timelike invariants, the last being responsible of non-syncronization.

In such a generalized field theory, the Einstein's equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \varkappa T_{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4 \tag{33}$$

can be used only to give the structure of the energy-momentum tensor [4].

5. A Verification of the Model. Some interesting consequences result from the above model.

(i) The Lorentz transformations

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}, \quad t = \frac{t' + x'v/c^2}{\sqrt{1 - v^2/c^2}} \tag{34}$$

determine a subgroup of the group (7) by apolar transport. Indeed, if the condition (15) is used, then the infinitesimal generators of the group (7) have the form [2]

$$\hat{X}_1 = \tfrac{1}{2}(u^2 - v^2)\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v}; \quad \hat{X}_2 = \tfrac{1}{2}\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right); \quad \hat{X}_3 = \frac{\partial}{\partial u}. \tag{35}$$

A subgroup of this group can be obtained if we take $u = v$. Its infinitesimal generators become then

$$\hat{X}_1 = u^2 \frac{\partial}{\partial u}; \quad \hat{X}_2 = u\frac{\partial}{\partial u}; \quad \hat{X}_3 = \frac{\partial}{\partial u} \tag{36}$$

and they span the homography group with a variable

$$u' = \frac{Au + B}{Cu + D}. \tag{37}$$

If

$$A = D = \cosh\alpha; \quad B = C = \sinh\alpha; \quad \tanh = \frac{v}{c}, \quad u' = \frac{x}{ct}; \quad u = \frac{x'}{ct'}; \tag{38}$$

then from (37) it results again (34).

6. Conclusions. In our opinion, the above model is the simplest and natural model of gravitation. The gravitational field appears as an intrinsic pro-

perty of the particle spin. Because the SU_2 algebra determines the direction cosines of the physical frame, we belive that this model can be used as a started point for a quantum theory of gravitation.

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O TEORIE A GRAVITAŢIEI BAZATĂ PE CONCEPTUL DE REPER FIZIC

(Rezumat)

Se construieşte un reper ortogonal ca soluţie a unei cubice binare. Condiţia de apolaritate selectează reperul fizic din clasa reperelor matematice. În aceste condiţii reperul fizic este definit cu ajutorul parametrilor de spin.

Principiul variaţional este aplicat în ipoteza existenţei unui paralelism de direcţii. Acesta permite construcţia metricilor (statice, staţionare) independent de ecuaţiile lui Einstein.

În acest model gravitaţia apare natural, ca o proprietate intrinsecă a spinului particulelor.

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LA DYNAMIQUE DU SYSTÈME SOLAIRE COMME UN ARGUMENT SCIENTIFIQUE DE LA SYMBOLISTIQUE YIN-YANG (I)

PAR

I. VULPOI-NAGHEL et MARCEL AGOP

0. Introduction. En utilisant la théorie des invariants adiabatiques et les faits de l'astronomie on montre que la dynamique du système Terra-Soleil (T—S) este dominée par l'expansion de Hubble.

Si on étudie le mouvement d'une particule dans un potentiel combiné attractif $\left(\sim -\frac{1}{r}\right)$ — répulsif ($\sim r^2$) la trajectoire s'exprime par une fonction elliptique ce qui peut expliquer l'avance du périhélie des planètes. Le mouvement attractif s'obtient comme une dégénérescence de la fonction elliptique dans la période imaginaire et le mouvement répulsif comme une solution solitonique. En appliquant ce modèle à l'interaction (T—S) résulte que dans l'étape actuelle la dynamique du ce système est dominée par la répulsion.

1. La dynamique du système (T—S). La combustion nucléaire et la contraction gravitationelle sont les processus qui font que les masses des étoiles ne restent pas constantes en temps : elles varient plus lentement comme les étoiles du type Soleil, ou rapidement comme les étoiles doubles [1], [2]. Dans ce contexte le système (T—S) fonctionne avec la diminution lente en temps de sa masse totale.

Vraiment, en considérant le Soleil comme une étoile qui a la luminosité constante [1], [2]

$$L = 3{,}8.10^{26} \quad \text{W}, \tag{1}$$

la perte de masse par radiation dans l'unité de temps, étant le principal processus qui conduit à la diminution de sa masse, aura la valeur

$$\left(\frac{\Delta m}{\Delta t}\right)_S = \frac{L}{c^2} \approx 4.10^9 \quad \text{kg/s}. \tag{2}$$

En ce qui concerne la variation de la masse de Terra elle est donnée en principal par le rapport entre l'accumulation de poussière météorique et la perte d'énergie par radiation thermique. Comme le premier processus a la valeur [1], [2]

$$\left(\frac{\Delta m_1}{\Delta t}\right) \approx 10^2 \quad \text{kg/s} \tag{3}$$

et la perte d'énergie par radiation thermique est

$$\frac{\Delta m_2}{\Delta t} = \frac{\sigma s T^4}{c^2} \approx 10^{-2} \quad \text{kg/s}, \tag{4}$$

avec

$$\sigma = 5{,}68.10^{-8}\ \text{J/m}^2\text{K}^2\ ; \quad R \approx 6{,}4.10^6\ \text{m}\ ; \quad T_0 \approx 3.10^2\ \text{K},$$

où on considère que R=constante, résulte que le bilan massique pour Terra est positif

$$\left(\frac{\Delta m}{\Delta t}\right)_T = \frac{\Delta m_1}{\Delta t} - \frac{\Delta m_2}{\Delta t} \approx 10^2 \quad \text{kg/s}, \tag{5}$$

ce qui montre que la masse de T augmente en temps (la diminution de la vitesse de rotation de Terra soutient cette afirmation [1]). En ce mode la tranche de diminution de la masse totale pour le système (T—S) est

$$\left(\frac{\Delta m}{\Delta t}\right)_{S-T} \approx 10^9 \quad \text{kg/s}. \tag{6}$$

En sachant maintenaint que la masse totale du système est

$$M = m_S + m_T \approx 10^{30} \quad \text{kg}, \tag{7}$$

il en résulte que le temps dans lequel cette masse devient nulle est

$$T = \frac{\Delta M}{\left(\frac{\Delta m}{\Delta t}\right)_{S-T}} \approx 10^{20} \quad \text{s}, \tag{7'}$$

valeur en accord avec les données fournisées par les investigations géologiques [1], [2].

Une autre conclusion intéressante donnée par cette analyse est que la masse du système (T—S) peut être considérée comme un paramètre adiabatique en raport avec l'échelle de temps géologique. Dans ces conditions on montre que le produit de la masse totale et la distance relative des deux corps reste constante [3]

$$M(t)r = \text{const.} \tag{8}$$

Alors, en partant de la relation (8) il résulte que pour le système (T—S) la distance relative entre ces corps augmentera. Il existe dans ce sens une série d'arguments: les planètes gigantes sont situées en l'intérieur du système solaire, le diamètre aparent du S vu de T diminue constament [1], [2].

Les résultats antérieurs indiquent donc que notre système solaire est en expansion. Comme l'unique processus observable actuellement au niveau de la Métagalaxie est l'effet Hubble il résulte que le même effet existe au niveau de système solaire — la valeur 10^{20} s que l'on a obtenue comme temps géologique pour le système (T—S) est égale avec la constante de Hubble de la loi[1]

$$v = Hr\ ; \quad H = \text{const.}, \tag{9}$$

ou v est la vitesse d'expansion des Galaxies et r —leur distance relative. Dans cettes conditions un calcul très simple indique que T s'éloigne de S avec une ratte annuelle dè

$$v \approx 10^{-2} \quad \text{mm/année.} \tag{10}$$

Nous étudions maintenant le mouvement du système S—T dans le but d'expliquer cette expansion. En ce contexte on montre que l'adjonction d'une force répulsive du tipe $\mathbf{F}=m_T\mathbf{r}/T^2$ avec $H=1/T$ à la force newtonienne peut expliquer le mouvement rappelé. Soit alors le potentiel

$$V=-G\frac{m_S m_T}{r}+\frac{m_T r^2}{T^2}\,; \quad G,\ T=\text{const.} \tag{11}$$

Pour ce potentiel les lois de l'énergie et du moment cinétique ont les expressions

$$v^2=\frac{2E}{m_T}+\frac{2Gm_S}{r}-\frac{2r^2}{T^2}\,; \quad E=\text{const.}, \tag{12}$$

et respectivement

$$r^2\dot{\theta}=\frac{L}{m_T}\,; \quad L=\text{const.} \tag{13}$$

L'équation de la trajectoire s'obtient par l'élimination du temps entre les relations (12) et (13). En ce but en partant de la relation (12) sur la forme

$$\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2+r^2\right]\dot{\theta}^2=\frac{2E}{m_T}+\frac{2Gm_S}{r}-\frac{2r^2}{T^2}\,, \tag{14}$$

utilisant (13) et la substitusion

$$u=\frac{n}{r}\,; \quad n=\frac{2Gm_S}{c^2}\,, \tag{15}$$

l'équation diférentielle de la trajectorie est

$$u^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2=-u^4+\left(\frac{ncm_T}{L}\right)^2u^3+\frac{2Em_Tn^2}{L^2}u^2-\frac{2m_T^2n^4}{L^2T^2}\,. \tag{16}$$

L'intégration de (16) donne

$$\theta=\pm\int\frac{u\mathrm{d}u}{P(u)}+\text{const.}, \tag{17}$$

avec

$$P(u)=-u^4+\left(\frac{nm_Tc}{L}\right)^2u^3+\frac{2Em_Tn^2}{L^2}u^2-\frac{2m_Tn^4}{L^2T^2}\,, \tag{18}$$

c'est à dire une intégrale elliptique de la deuxième expèce.

Soit e_1, e_2, e_3, e_4 les racines réelles du polynomme (18) avec

$$e_1<e_2<e_3<e_4\,; \tag{19}$$

alors, le changement de variable

$$u = \frac{e_1 - e_2 s^2}{1 - s^2} \tag{20}$$

transforme l'intégrale (17) dans

$$\theta = \pm \frac{2}{\sqrt{(e_3 - e_2)(e_4 - e_2)}} \int \frac{(e_1 - e_2 s^2)\mathrm{d}s}{(1 - s^2)\sqrt{(b_M^2 - s^2)(s^2 - b_m^2)}} + \text{const.}, \tag{21}$$

où

$$b_m^2 = \frac{e_3 - e_1}{e_3 - e_2}, \quad b_M^2 = \frac{e_4 - e_1}{e_4 - e_2}. \tag{22}$$

Avec

$$s = b_M \mathrm{dn}\, \tau \tag{23}$$

l'intégrale (21) devient

$$\theta = \pm \left[\frac{2e_3}{\sqrt{(e_3 - e_2)(e_4 - e_1)}} \int \mathrm{d}\tau + \frac{2}{\sqrt{(e_4 - e_1)(e_3 - e_2)}} \int \frac{(e_1 - e_2)\,\mathrm{d}\tau}{1 - b_M^2 \mathrm{dn}^2 \tau} \right] + \text{const.}, \tag{24}$$

où on a utilisé les propriétés des fonctions elliptiques [4]:

$$\begin{gathered} \mathrm{sn}^2\tau + \mathrm{cn}^2\tau = 1\,; \quad \mathrm{dn}^2\tau + k^2 \mathrm{sn}^2\tau = 1, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{dn}\tau) = -k^2\, \mathrm{sn}\tau\, \mathrm{cn}\tau; \quad k^2 = 1 - \frac{b_m^2}{b_M^2}. \end{gathered} \tag{25}$$

Alors s est une fonction elliptique de τ avec les périodes

$$\omega_1 = 4 \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}\,; \quad \omega_2 = 2\mathrm{i} \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}, \tag{26}$$

où

$$k^2 + k'^2 = 1\,; \quad s = \sin\varphi. \tag{27}$$

Si r varie de $(b_M^2 - 1)/(e_2 b_M^2 - e_1)$ à $(b_m^{21} - 1)/(e_2 b_m^2 - e_1)$ et de retour, θ varie avec la quantité approximative

$$\delta\theta = \frac{2\pi}{\sqrt{(e_4 - e_1)(e_3 - e_2)}} [1 + \tfrac{1}{4}(1 - k^2)]. \tag{28}$$

On met ainsi en évidence le fait q'une force répulsive pêut produire au niveau de notre système solaire un "avance du périhelie", phénomène observable ayant comme effet la non fermeture de la trajectoire planétaire. Donc, dans l'étape actuelle d'évolution du système solaire, sont prépondérantes les forces répulsives.

Les cases clasiques du mouvement de T autour du soleil (mouvement d'atraction ou pur répulsif) s'obtiennent aux limites asiymptotiques des periodes ω_1 et ω_2. Ainsi, si $\omega_2 \to \infty$ alors entre ces deux corps s'exerce seulement des forces de type atractif. En ce cas l'intégrale (17) a la forme

$$\theta = \pm \int \frac{du}{\sqrt{-u^2 + \frac{2Gm_S m_T}{L^2} u + \frac{2Em_T}{L^2}}} + \text{const.}, \quad u = \frac{1}{r}. \tag{29}$$

Si les racines du polynome sous le radical de (29) sont réelles, c'est-à-dire

$$\frac{G^2 m_S^2 m_T^3}{2L^2} + E > 0, \tag{30}$$

alors elles ont les expressions :

$$\begin{aligned} u_1 &= \frac{Gm_S m_T^2}{L^2}\left[1 + \left(1 + \frac{2EL^2}{G^2 m_S^2 m_T^3}\right)^{1/2}\right], \\ u_2 &= \frac{Gm_S m_T^2}{L^2}\left[1 - \left(1 + \frac{2EL^2}{G^2 m_S^2 m_T^3}\right)^{1/2}\right]. \end{aligned} \tag{31}$$

Maintenant, la substitution

$$[(u-u_1)(u_2-u)]^{1/2} = t(u-u_1) \tag{32}$$

transforme l'intégrale (29) dans

$$\theta = \pm 2 \int \frac{dt}{t^2+1} + \text{const.}; \tag{33}$$

avec le signe + devant la relation (33) il vient que

$$\frac{\theta}{2} + \text{const.} = \text{arctg}\, t. \tag{34}$$

Après un calcul relatif simple, on obtient la section conique

$$\frac{1}{r} = \frac{1 + c\cos(\theta + \theta')}{p}, \tag{35}$$

avec

$$p = \frac{L^2}{Gm_S m_T^2}; \quad e = \left(1 + \frac{2EL^2}{G^2 m_S^2 m_T^3}\right)^{1/2}; \quad \theta' = \text{const.} \tag{36}$$

On observe que pour $E<0$ la trajectoire est une ellipse; si $E>0$, la trajectoire est une hyperbole et $E=0$ implique que la trajectoire est une parabole.

Le cas $\omega_1 \to \infty$, conduisant à l'existence des forces répulsives, est la solution solitonique [5]. En ce cas, par un raisonnement similaire on peut montrer que la trajectoire se réduit à la forme

$$z = \frac{1}{p'}[1 + e'\text{ch}(\theta + \theta_0)], \tag{37}$$

où

$$z = \frac{1}{r^2}; \quad p' = \frac{L^2}{Em_T}; \quad e' = \left(\frac{2L^2}{E^2 T^2} - 1\right)^{1/2}; \quad \theta_0 = \text{const.} \tag{38}$$

Donc, ce modèle peut expliquer la dynamique actuelle du système solaire.

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DINAMICA SISTEMULUI SOLAR CA ARGUMENT ŞTIINŢIFIC AL SIMBOLISTICII YIN-YANG (I)

(Rezumat)

Utilizînd teoria invarianţilor adiabatici şi datele furnizate de astronomie se arată că dinamica sistemului Soare—Pămînt este dominată de expansiunea Hubble.

Acceptînd existenţa la nivelul sistemului solar a două tipuri de forţe — una atractivă ($\sim 1/r$) şi alta repulsivă ($\sim r^2$) se poate explica într-un asemenea context atît expansiunea Hubble cît şi avansul periheliului planetar. Cazurile clasice ale mişcării se obţin ca situaţii asimptotice în perioadele unei anumite funcţii eliptice — de exemplu expansiunea Hubble poate corespunde soluţiei solitonice a problemei.

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THE PHONON-PHONON DRAG IN SEMICONDUCTOR WITH POSITION-DEPENDENT BAND STRUCTURE

BY

MARIANA ISTRATE, D. SCUTARU and SORIN ISTRATE

1. Introduction. The electron-phonon interaction is of a particular importance when one of two systems is not at equilibrium. The nonequilibrium of the electron or phonon system is induced by the external fields (an electric field, a temperature gradient s.o.). The application of an electric field to a metal or semiconductors leads to the deviations of the electron (and holes) distribution function from its equilibrium value, and the carriers acquire a drift velocity. The phonon distribution is also modified through the electron-phonon interaction and the phonons acquires a drift velocity parallel to that of the carriers, the efect being called the carriers drag of phonons. To the application of a temperature gradient, the phonon acquires a drift velocity. The carriers distribution is also modified through the carrier-phonon interaction and the carriers acquires a drift velocity parallel to that of the phonons, the effect being called the phonon drag of carriers [1].

In the semiconductors, the electron-phonon interaction is elastic if the phonon wave vector **p** satisfy the inequality $p<2k$, where **k** is the electron wave vector [2]. This phonons are called the electron phonons. The phonons for which $p>2k$ are called the thermic phonons. In semiconductor can appear thus a drag in two steps. To the application of a temperature gradient the thermic phonons drag the electron-phonons (which gives a supplementary drift velocity) and the electron-phonons drag the carriers. To the application of an electric field the carriers drag the electron-phonons and the electron-phonons drag the thermic-phonons.

In this paper we report a theoretical study on the effect of drag in two steps on some kinetic coefficients in a semiconductor with position-dependence band structure, of n type. A graded composition of two or more components, a dependence on position of the impurity doping, a nonuniform strain or a nonuniform temperature determine a position-dependent band structure. If a semiconductor has a position-dependent lattice structure, it also has a position dependent band gap E_g, bottom of the conduction band E_c, top of the valence band E_v, density of states g and electron affinity χ [3]...[6].

2. The Boltzmann Equations. The electrons, electron-phonon and thermic-phonon distribution at nonequilibrium are

$$f(\mathbf{k}, \mathbf{r})=f_0(\mathbf{k}, \mathbf{r})+k_T f_1^*(\mathbf{k}, \mathbf{r}) \quad \text{(electron)}, \tag{1}$$

$$G(\mathbf{p}, \mathbf{r}) = G_0(\mathbf{p}, \mathbf{r}) + p_T G_1^*(\mathbf{p}, \mathbf{r}) \quad \text{(electron-phonon)}, \tag{2}$$

$$F(\mathbf{q}, \mathbf{r}) = F_0(\mathbf{q}, \mathbf{r}) + q_T F_1^*(\mathbf{q}, \mathbf{r}) \quad \text{(thermic-phonon)}, \tag{3}$$

where $f_0(\mathbf{k}, \mathbf{r})$, $G_0(\mathbf{p}, \mathbf{r})$, $F_0(\mathbf{q}, \mathbf{r})$ are the isotropic part of the distribution functions and k_T, p_T, q_T are the components of wave vector in the direction of the external field. The Boltzmann equations for the electrons, electron-phonons and thermic-phonons taking into account the drag in two steps, in the semiconductors with position band-structure are :

$$\nabla f_0 - \frac{1}{\hbar}(\nabla E_c + \nabla W)\nabla_k f_0 = I_{ep}\{f, G\} + I_{ed}\{f\}, \tag{4}$$

$$s\nabla T \frac{\partial G_0}{\partial T} = I_{pe}\{G, f\} + I_{pT}\{G, F\} + I_{pd}\{G\}, \tag{5}$$

$$s\nabla T \frac{\partial F_0}{\partial T} = I_{Tp}\{F, G\} + I_{TT}\{F\} + I_{Td}\{F\}, \tag{6}$$

where I_{ep}, I_{ed}, I_{pe}, I_{pT}, I_{pd}, I_{Tp}, I_{TT}, I_{Td} are the electron-electron phonon scattering integral, electron-defect scattering integral, electron phonon-eclectron scattering integral s.o., W is the kinetic energy of the electron and s the longitudinal sound velocity.

In our notations, $G_1^*(\mathbf{k}, \mathbf{r})$ and $F_1^*(\mathbf{q}, \mathbf{r})$ are given by the expresions [6], [7]

$$G_1^*(\mathbf{k}, \mathbf{r}) = -G_0(G_0+1)\frac{1}{k_0 T}\left[L_{pe}^{-1}L_p \int_{F(p/2)}^{\infty} \mathbf{V}_E \frac{\partial f_0}{\partial E}\, dE + L_{pT}^{-1}L_p \mathbf{U} - sL_p T^{-1}\nabla T\right]. \tag{7}$$

$$F_1^*(\mathbf{q}, \mathbf{r}) = -F_0(F_0+1)\frac{1}{k_B T}\left[L_{Tp}^{-1}L_T \langle\langle \mathbf{U}_{(p)} \rangle\rangle_n - sL_T T^{-1}\nabla T\right], \tag{8}$$

where $L_T^{-1} = L_{Tp}^{-1} + L_{Td}^{-1}$, $L_p^{-1} = L_{pe}^{-1} + L_{pT}^{-1} + L_{pd}^{-1}$ and L_{pe}, L_{pT}, L_{pd} s.o. are the mean free paths. The interactions with defects and with electron-phonons are given by the relations [1]

$$I_{ed}\{f\} = -\frac{k_T f_1^*(\mathbf{k}, \mathbf{r})}{\tau_{ed}}, \tag{9}$$

$$\begin{aligned} I_{ep} = D \sum_p \frac{\omega_p}{c} \Big[&\{[1 - f(\mathbf{k}, \mathbf{r})] f(\mathbf{k}+\mathbf{p}, \mathbf{r})[G(\mathbf{p}, \mathbf{r}) + 1] - \\ &- [1 - f(\mathbf{k}+\mathbf{p}, \mathbf{r})] f(\mathbf{k}, \mathbf{r}) G(\mathbf{p}, \mathbf{r})\} \delta[E(\mathbf{k}+\mathbf{p}, \mathbf{r}) - E(\mathbf{k}, \mathbf{r}) - \hbar\omega_q] + \\ &+ \{[1 - f(\mathbf{k}, \mathbf{r})] f(\mathbf{k}-\mathbf{p}, \mathbf{r}) G(\mathbf{p}, \mathbf{r}) - [1 - f(\mathbf{k}-\mathbf{p}, \mathbf{r})] f(\mathbf{k}, \mathbf{r})[G(\mathbf{p}, \mathbf{r}) + 1]\} \cdot \\ &\cdot \delta[E(\mathbf{k}-\mathbf{p}, \mathbf{r}) - E(\mathbf{k}, \mathbf{r}) + \hbar\omega_q]; \qquad D = \frac{\pi C_1^2}{V\rho c}, \end{aligned} \tag{10}$$

C_1 is the total deformation potential constant.

In the degenerate semiconductors, the equilibrium distribution function is

$$f_0(\mathbf{k}, \mathbf{r}) = \left[e^{\frac{E(\mathbf{k}, \mathbf{r}) - E_{Fn}(\mathbf{r})}{k_B T}} + 1 \right]^{-1}, \tag{11}$$

where E_{Fn} is the quasi-Fermi level. Introducing the relations (1) and (2) in (9), retaining only the linear terms in k_T, p_T (the method presented in [9]), calculating the integrals and using the relations (10), we obtain for the anisotropic part of the distribution function the expresion

$$k_T f_1^*(\mathbf{k}, \mathbf{r}) = \tau \mathbf{V}_k \nabla E_{Fn} \frac{\partial f_0(\mathbf{k}, \mathbf{r})}{\partial E} (1 + D) + \tau \mathbf{V}_k \frac{E - E_{Fn}}{T} \nabla T \frac{\partial f_0}{\partial E}, \tag{12}$$

where

$$\frac{1}{\tau} = \frac{1}{\tau_a} + \frac{1}{\tau_{ed}}, \tag{13}$$

$$D = \frac{k_T f_0(\mathbf{k}, \mathbf{r}) [1 - f_0(\mathbf{k}, \mathbf{r})]}{x \mathbf{V}_k \nabla E_{Fn} \frac{\partial f_0(\mathbf{k}, \mathbf{r})}{\partial E}}, \tag{14}$$

$$\frac{1}{x} = \frac{C_1^2 m^*}{\rho \hbar k_B T_c} \int_0^{2k} G_1^*(\mathbf{p}, \mathbf{r}) p^5 \mathrm{d}p; \tag{15}$$

τ_a is the electron-electron phonon relaxation time when the anisotropic part of the distribution function is neglected.

3. The Transport Coefficients. Introducing (12) in the density of current expression

$$\mathbf{j} = -\frac{e}{4\pi^3} \int \mathbf{V}_k k_T f_1^*(\mathbf{k}, \mathbf{r}) \mathrm{d}^3 k, \tag{16}$$

we have obtained

$$\mathbf{j} = n \mu_n \nabla E_{Fn} + L_n \frac{\nabla T}{T}; \tag{17}$$

μ_n is the electron mobility when we take into account the drag in two steps

$$\mu_n = -\frac{e}{4\pi^3 n} \int \tau \mathbf{V}_k^2 \frac{\partial f_0(\mathbf{k}, \mathbf{r})}{\partial E} (1 + D) \mathrm{d}^3 k \tag{18}$$

and the transport coefficient L_n is

$$L_n = \frac{e}{4\pi^3} \int \tau \mathbf{V}_k^2 (E - E_{Fn}) \frac{\partial f_0(\mathbf{k}, \mathbf{r})}{\partial E} \mathrm{d}^3 k. \tag{19}$$

In the semiconductors with position band structure, the quasi Fermi level

is given by the relations [4]

$$\nabla E_{Fn}=\nabla E_c+\left(\frac{\partial E_{Fn}}{\partial n}\right)_{T,\varepsilon}\nabla n+\left(\frac{\partial E_{Fn}}{\partial T}\right)_{n,\varepsilon}\nabla T-$$

(20)

$$-\left(\frac{\partial E_{Fn}}{\partial n}\right)_{T,\varepsilon}\frac{1}{4\pi^3}\int \mathrm{d}^3k\frac{\partial f_0(\mathbf{k},\mathbf{r})}{\partial E}\nabla W,$$

and

(21) $$E_0-e\varphi-\chi-E_g=E_c-E_g\,.$$

Using (20), (21) in (16), from the usual condition $j=0$, we have obtained the differential thermoelectric power (the Seebeck coefficient)

$$\alpha=-\frac{1}{n\mu_n}\left[n\mu_n\left(\frac{\partial E_{Fn}}{\partial T}\right)_{n,\varepsilon}+n\mu_n\left(\frac{\partial E_{Fn}}{\partial n}\right)_{T,\varepsilon}\frac{\partial n}{\partial T}-\right.$$

(22)

$$\left.-n\mu_n\left(\frac{\partial E_{Fn}}{\partial n}\right)\frac{1}{4\pi^3}\int \mathrm{d}^3k\,\frac{\partial f_0(\mathbf{k},\mathbf{r})}{\partial E}\frac{\partial W}{\partial T}-n\mu_n\frac{\partial\chi}{\partial T}+\frac{L_n}{T}\right].$$

The density of the energy flux transported by the electrons is given by

(23) $$\mathbf{w}=\int f_1(\mathbf{k},\mathbf{r})[E(\mathbf{k},\mathbf{r})-e\varphi]\mathbf{V}_k\frac{\mathrm{d}^3k}{4\pi^3}\;;$$

φ is the electrostatic potential. Using (10) and (17) it results

(24) $$\mathbf{w}=\varphi\mathbf{j}+\pi\mathbf{j}+\varkappa\nabla T,$$

where the Peltier coefficient is

(25) $$\pi=\int\tau\mathbf{V}_k\frac{\partial f_0(\mathbf{k},\mathbf{r})}{\partial E}(1+D)\frac{\mathbf{V}_k}{n\mu_n}E(\mathbf{k},\mathbf{r})\frac{\mathrm{d}^3k}{4\pi^3}$$

and the heat conductivity coefficient is

(26) $$\varkappa=\int\tau\mathbf{V}_k^2\frac{\partial f_0(\mathbf{k},\mathbf{r})}{\partial E}\frac{E(\mathbf{k},\mathbf{r})}{4\pi^3}\left[\frac{(1+D)L_{n1}}{Tn\mu_n}+\frac{E-E_{Fn}}{T}\right]\mathrm{d}^3k.$$

4. **Conclusions**,. The relations (18), (19), (25) and (26) are the general expressions for the transport coefficients in the nonhomogeneous semiconductor of n-type, if we take into account the drag in two steps. The drag effect modifies the transport coefficients. For example if the drag effect is neglected, $G_1^*=0$ and from (15) and (12) results that $D=0$. In this case we reobtain from (18) the expression which have been calculated in [4]. For a concrete calculation we must know the position dependent E_c, E_v, E_{Fn}, χ and g.

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ANTRENAREA FONON-FONON ÎN SEMICONDUCTORI CU STRUCTURA DE BANDĂ DEPENDENTĂ DE POZIŢIE

(Rezumat)

Se analizează fenomenul de antrenare în două trepte, în semiconductori cu structură de bandă dependentă de poziţie, de tip *n*. Se rezolvă ecuaţiile Boltzmann pentru electroni, fononi electronici şi fononi termici şi se obţin expresiile coeficienţilor de transport.

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Prof. dr. Ion I. Enescu

1930—1989

It was the last day of March 1989 when the Department of Mathematics, the whole Polytechnic Institute of Jassy and the entire higher education community of our city suffered a heavy and irretrievable loss : Professor Dr. Ion Enescu — a distinguished and beloved educator, an outstanding mathematician, a man of a rare and noble spiritual conduct — suddenly and unexpectedly died.

Professor Ion Enescu was born on June 14, 1930 in Păţeşti — Focşani, a village in Vrancea Country. His parents — Ion and Emilia Enescu — were teachers of Romanian and French, respectively, in Focşani. No wonder that he was educated, from his early years, in the spirit of deference to the great values of national and world culture. I. Enescu attended his secondary studies at the famous Lycée "Unirea" of Focşani. The skilled staff of this prestigious school helped him in assimilating an even broader and solid general culture. Although the humanistic of the education received at home was naturally prevailing, when he was to start his higher studies he chose the exact sciences : more precisely, *mathematics*. He attended his higher studies at the prestigious Faculty of Mathematics and Physics of the "Al. I. Cuza" University of Jassy

(section of Mathematics), between 1949 and 1953. He had thus the privilege to be a student of the great professors who brought national and worldwide recognition to the Jassy school of mathematics : academicians Alexandru Myller, Octav Mayer and Mendel Haimovici, professors Ion Creangă, Gheorghe Gheorghiev, Adolf Haimovici and Ilie Popa. As remarked by Professor I. Creangă in his report on I. Enescu's activity (on the occasion of a contest that was to promote him to the position of professor), even since a student he proved much interest, passion and skill for research work in mathematics ; it was just then possible to foresee his capability and gift for personal creative work in the field of mathematics. I. Enescu actually was one of the best students of his series. It should be said that several graduates of this series became prominent representatives of Romanian mathematics since the sixties. Quite naturally he was recommended to be taken on the university staff.

After a first year of activity as an instructor at the Railway Institute of Bucharest (1953—1954), he was taken on the staff of the Department of Mathematics — Polytechnic Institute of Jassy. It was amidst this collective that he was to carry on a prodigious didactic and scientific activity during 35 years, firmly supported by un unbounded self-abnegation, by a native teaching gift, by an uncompromising self-exigency. It was therefore absolutely normal that he should have climbed, quickly and naturally, all the steps of an exemplary university career : instructor (1953—1961), assistant professor (1961—1969), associate professor (1969—1975), and professor (1975—1989). Not less than 28 series of students, that is thousands of the existing Romanian engineers, have therefore had the opportunity to appreciate and to admire the rigour, the clarity and accessibility of all the courses given by him. The same exquisite didactic qualities may be recaptured by reading the texts of his courses published by the Polytechnic Institute of Jassy : *Analytical and Differential Geometry* — 1973, *Special Mathematics* (in cooperation with V. Sava) — 1981. A brilliant professor for so many series of students, I. Enescu has been — no less — *a model of professional scrupulousness and competence* to all his colleagues from the Department of Mathematics, from the Faculty of Mechanics and from the Institute itself.

Professor I. Enescu's *scientific activity* was not less outstanding than the didactic one. It was focused on some problems of the *Theory of Algebras* and it materialized in the monograph *Algebras* (in cooperation with I. Creangă, Technical Publ. House — Bucharest 1973) and 35 papers published in mathematical journals. His PhD thesis also brought important contributions to the theory of algebras. Prof. I. Enescu's research pursuits and original results may be grouped within the following six main topics :

1. The study of some classes of real associative commutative finite order algebras.
2. The study of associative commutative algebras of order five, with identity element and real scalars.
3. Finite order algebras over a noncommutative field.
4. Associative noncommutative algebras.

5. Nonassociative algebras.

6. Applications of some of the preceding algebraic structures in mechanics, in the study of linear oscillations, and in the kinetics of chemical reactions.

It would be a hard task to describe the original results due to Professor Enescu within a couple of pages. We shall however try to do it in brief : within topic 1, I. Enescu studied the polynomial algebras with real scalars, obtaining some results which determined necessary and sufficient conditions for the polynomiality of an algebra together with its structural properties ; the following papers are representative in this respect : *On the Polynomial Algebras* (I), (II) (Bulletin of the Polytechnic Institute of Jassy, 1965 and 1966). He then built up classes of algebras of order $4n$ and $8n$, respectively, by considering ordered pairs of real quaternions and defining their multiplication, in papers like *Sur une classe d'algèbres d'ordre 8n* and *Sur une classe d'algèbres d'ordre 4n* (Bull. I.P.I. — 1968). He also determined the nonisomorphic types of nilpotent algebras of order five for a double-parametric family (Bull. I.P.I. — 1973) while — within Topic 2 — he found out a constructive procedure for the exhaustive determination of all the nonisomorphic types of real, associative and commutative algebras of order five with identity (Bull. I.P.I. — 1967). Topic 3 is best illustrated by the paper *Algèbres associatives sur un corps noncommutatif* (Bull. I.P.I. — 1969), where he built up a model of algebra over a noncommutative field, starting from the idea to immerse a commutative ring into a noncommutative one. Within Topic 4 he established some properties of the semisimple noncommutative algebras — in *On Semisimple Noncommutative Algebras of Finite Order* (Bull. I.P.I. — 1970) — which he then employed in the study of the reversive algebras (Bull. I.P.I. — 1970). As regards Topic 5, we quote *On Some Families of Algebras of Order Three* (Bull. I.P.I. — 1974), in which he studied the families of nonassociative algebras of order three corresponding to the system of differential equations which describe the kinetics of irreversive reactions between three chemical species. By the way, I. Enescu's concerns regarding the nonassociative algebras were consistently represented by other 6 papers, published between 1975 and 1988. The relevance of the study of algebras to other fields of science and technique is proved by his papers falling within Topic 6. For instance, in *On the Polynomial Algebra Character of Some Hypercomplex Numbers Used in the Study of Some Linear Oscillations* (Bull. I.P.I. — 1966), Prof. I. Enescu showed that certain such hypercomplex numbers used to describe linear vibrations subjected to some restrictions of physical nature may be naturally structured as polynomial algebras. In other two papers (published in Bull. I.P.I. — 1972 and 1973, in cooperation), he studied the chemical algebras associated to the system of differential equations governing the consecutive chemical reactions and other types of reactions; the results thus established allow to classify these reactions and to know certain of their properties.

Part of the results of Professor Enescu's intensive research work on the algebras were included and developed in Chapters 7—11 of the

monograph *Algebras* (previously quoted), as well as in his doctoral thesis entitled *Contributions to the Study of Real Polynomial Algebras, Extensions and Applications,* presented at the Faculty of Mathematics — "Al. I. Cuza" University of Jassy, in 1967.

Thanks to the whole of his scientific work, Prof. Ion Enescu asserted himself as a deep algebraist, as the most representative and outstanding continuer of the Jassy school of research on the Theory of Algebras founded by Professor Ion Creangă. By the way, most of the papers written by I. Enescu were reviewed in the well-known review journals "Referativnyj žurnal" (USSR) and "Mathematical Reviews" (USA). I. Enescu himself has been an active reviewer for "Mathematical Reviews" since 1971. His results were mentioned and used in several textbooks and monographs by Romanian authors such as professors I. Creangă, M. Roşculeţ, Ion D. Ion, A. Corduneanu, Academician Caius Iacob and many others. Four Romanian specialists (A. Braier, C. Boţan, V. Tamaş, P. Izvercianu) made use of his results in their PhD theses. Many foreign mathematicians asked him for papers and used them : E. W. Wallace (Univ. of Leeds), Earl Taft (Rutgers, The State University, New Brunswick, N. J.), V. V. Višnevskij (Univ. Kazan — USSR), Robert Melter (Southampton College, N.Y.), B. Roux (Faculté des Sciences — Univ. de Montpellier), J. H. W. Hurt (Dept. of Mathematics, Univ. of Saskatchewan, USA) and others. We cannot mention here the many Romanian mathematicians who used and developed his results.

Between 1982 and 1989, Professor I. Enescu was the Head of the Department of Mathematics. This has been an essential component of his activity, to which he devoted himself with a lot of passion, competence and gift. All the successes achieved by our department during these seven years have been due to him in most part. In any situation, he was fighting for the good name of the department, for the dignity and respect deserved by our profession. He was doing it, without any compromise, until the very last moment of his life, entirely devoted to Science, to School, to Culture and to Human Ideals. He therefore was not only a great professor of the Polytechnic Institute of Jassy, but also an exceptional head of department.

But — above all — he was a wonderful colleague who deserves an ever-lasting remembrance from all of us. And his bright memory will certainly remain forever alive in our souls.

Department of Mathematics

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Lecturer dr. | Robert T. Vescan |

1951 —1989

Robert T. Vescan was born in Jassy on August 28, 1951. His father, Teofil Vescan, was an outstanding scientist and a distinguished professor of physics at the universities of both Cluj and Jassy. His mother, Agneta Vescan, was an associated professor of abstract algebra at the University „Al. I. Cuza" in Jassy.

Educated in an atmosphere of scientific emulation, he was attracted by the study of mathematics and his remarkable talent in this direction became soon evident. He finished his secondary school studies in 1970 with the highest possible scores at the Lycée—C. Negruzzi" in Jassy and, in the same year, he became a student at the Faculty of Mathematics and Mechanics at the University „Al. I. Cuza" in Jassy. He was a brilliant student and since 1971 he begun to perform original research in abstract algebra and analysis. He earned the licence in mathematics in 1974 and the Master Degree in Functional Equations in 1975, also with the highest possible scores. In this period (1971—1975) he wrote and published 4 papers. Since 1975 until 1979 he taught mathematics in the Energetic Lycée in Jassy and in 1979 he got an assistant professorship position at the Polytechnic Institute „Gh. Asachi" of Jassy. Since 1982 until his death he served as a lecturer of mathematics at the same institute.

His fields of interest were : *abstract algebra* (ordered structures, categories), *optimization* (Pareto minima), *Fourier analysis and non linear analysis* (variational and quasi-variational inequalities). He published 29 papers from which one jointly with I. Tofan and two jointly with J. Gottlieb and T. Jucan. On January 16, 1982, he earned the doctorate in mathematics from the University „Al. I. Cuza" in Jassy. His thesis entitled *Variational Inequalities*

with Applications to Infinite-dimensional Optimization written under the supervision of Professor Dr. Viorel Barbu is a remarkable contribution to the study of the existence and the uniqueness for quasi-variational inequalities (Q.V.I.) of abstract form. Roughly speaking, a Q.V.I. is an inequality involving a partial differential operator and whose solution must satisfy several additional *implicit constraints*. These inequalities have been introduced originally by A. Bensoussan, M. Goursat and J.-L. Lions (see C. R. Acad. Sci. Paris, *276* (1973), 1279–1284) in connection with some stochastic impulse control problems, and, in their most general form, by L. Tartar (see C. R. Acad. Sci. Paris, *278* (1975), 1193–1196) and by U. Mosco (see Lecture Notes in Math., *543* (1976), 83–156). The classical method for solving an inequality of this sort is either to construct a solution by an iteration procedure, or to prove the existence of a fixed point for a suitably defined multivalued mapping. It is the merit of Robert T. Vescan to develop a completely new and very subtle topological method of solving Q.V.I.'s based on some intersection results known as Knaster, Kuratowski, Mazurkiewicz's Lemma and Ky Fan's Lemma respectively. This method enabled him to obtain extremely sharp existence results referring to very general classes of Q.V.I.'s. See for instance his papers in: Rend. Sem. Mat. Univ. Padova, *63* (1980), 215–230; Numer. Funct. Anal. & Optimization, *2* (1980), 637–648; J. Math. Anal. & Appl., *93* (1983), 89–103; C. R. Acad. Sci. Paris, *299* (1984), 655–658. For a discussion of the main results of Robert T. Vescan see also the monograph of A. Bensoussan and J.-L. Lions *Inéquations variationnelles et quasi variationnelles en contrôle stochastique et en contrôle impulsionnel*, Dunod, Paris, 1982. We should mention here also his remarkable results on the nonlinear complementarity problem, as well as on Pareto minima.

Robert T. Vescan was a very fine speaker. His talks (communications, conferences, courses) were attended by many people, not only because of their high scientific level, but also for the quite personal manner of presentation. His style, both oral and in writings, can be characterized by clarity, simplicity, elegance and concision. He knew as nobody did how to acces in the most natural, simplest and convincing way the mind of his listeners and/or readers. He attended many national and international conferences, workshops and seminars in Iaşi, Bucureşti, Cluj-Napoca, Constanţa and Timişoara. His courses and seminary were very attractive and very much appreciated by the students. He was an extremely gifted educator.

After his unexpected and tragical death on August 9, 1989, Robert T. Vescan will be for many years to come remembered for his very deep achievements in mathematics, for his positive influence on students, colleagues and friends.

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